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ON THE ASYMPTOTICS FOR THE VACUUM EINSTEIN CONSTRAINT EQUATIONS

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Abstract

In this paper we prove density of asymptotically flat solutions with special asymptotics in general classes of solutions of the vacuum constraint equations. The first type of special asymptotic form we consider is called harmonic asymptotics. This generalizes in a natural way the conformally flat asymptotics for the $K = 0$ constraint equations. We show that solutions with harmonic asymptotics form a dense subset (in a suitable weighted Sobolev topology) of the full set of solutions. An important feature of this construction is that the approximation allows large changes in the angular momentum.

The second density theorem we prove allows us to approximate asymptotically flat initial data on a three-manifold M for the vacuum Einstein field equation by solutions which agree with the original data inside a given domain, and are identical to that of a suitable Kerr slice (or identical to a member of some other admissible family of solutions) outside a large ball in a given end. The construction generalizes work in [**C**], where the time-symmetric ($K = 0$) case was studied.

1. Introduction

We study the asymptotics of solutions to the vacuum constraint equations. In particular, we show how to approximate given initial data for the Einstein vacuum equation by data which is exactly that of a spacelike slice of a suitably chosen Kerr metric outside a compact set. For the time-symmetric case, the constraints reduce to the equation $R(g) = 0$, and it was shown in [**C**] how a Schwarzschild exterior can be selected and glued to asymptotically flat (AF) time-symmetric data and *compactly* perturbed to a solution of the time-symmetric constraint. In the non-time-symmetric case, one has to consider the second fundamental form, and hence linear and angular momentum of the data at infinity, for which the Kerr solution (for example) is aptly suited. We note that these constructions show that a compact piece of the data dictates (in some sense) only part of the asymptotic structure, namely that part

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which is governed by the energy-momentum and angular momentum. Having vacuum initial data which is identical to Kerr data outside a compact set is important for understanding the global evolution for the Einstein equations. In particular, such data evolves to produce a spacetime with particularly nice behavior at null infinity.

We briefly recall the basic setup of the constraint equations. A solution of the vacuum Einstein equation is a Lorentzian four-manifold $(\mathcal{S}, \bar{g})$ satisfying

$$\mathrm{Ric}\,(\bar{g}) - \frac{1}{2} R(\bar{g})\,\bar{g} = 0.$$

In this equation, $\mathrm{Ric}\,(\bar{g})$ and $R(\bar{g})$ denote the Ricci and scalar curvatures, respectively, of the metric $\bar{g}$. By taking the trace we see that this is equivalent to $\mathrm{Ric}\,(\bar{g}) = 0$. It is well-known that Einstein's equation admits an initial value formulation, in which the vacuum initial data consist of an oriented three-manifold M, a Riemannian metric g and a symmetric $(0,2)$-tensor K on M. The Gauss and Codazzi equations provide the constraints upon g and K in order that they form, respectively, the induced metric and second fundamental form of M inside a Ricci-flat spacetime $(\mathcal{S}, \bar{g})$; in the vacuum case these constraints are [**W**]

$$\begin{aligned} R(g) - |K|^2 + H^2 &= 0, \\ \mathrm{div}_{\,g}(K) - dH &= 0. \end{aligned}$$

Here $H = Tr_g(K) = g^{ij} K_{ij}$ denotes the mean curvature, $(\mathrm{div}_{\,g}(K))_i = g^{jk} K_{ij;k}$, and all quantities are computed with respect to g.

We rewrite these equations by introducing the momentum tensor

$$\pi^{ij} = K^{ij} - Tr_g(K) g^{ij}.$$

Note that we treat π as a tensor, following [**FM2**]; in the literature one also finds π defined as the *tensor density* $(K^{ij} - Tr_g(K) g^{ij})\sqrt{\det(g_{ij})}$. We also introduce functions $\mathcal{H}$ and Φ by

$$\begin{aligned} \mathcal{H}(g, \pi) &= R(g) + \frac{1}{2}\left(Tr_g \pi\right)^2 - |\pi|^2, \\ \Phi(g, \pi) &= \big(\mathcal{H}(g, \pi), \mathrm{div}_{\,g} \pi\big). \end{aligned}$$

The constraints then take the form $\Phi(g, \pi) = 0$.

Two well-known solutions of the vacuum Einstein equation are the Schwarzschild solution and the Kerr solution. The Schwarzschild metric is characterized by rotational symmetry and is *static*; indeed outside the horizon we have coordinates (t, x) in which the Schwarzschild metric takes the form ($r = |x|$)

$$g_S = -\left(1 - \frac{2m}{r}\right) dt^2 + \left(1 - \frac{2m}{r}\right)^{-1} dr^2 + r^2\, d\Omega^2.$$

In particular we see that the metric is static, as $\frac{\partial}{\partial t}$ is a timelike Killing field which is orthogonal to the time-slices. The Kerr solution is axisymmetric and stationary; the analogous Killing field is not orthogonal to the time-slices (physically, the black hole is rotating), and this can be easily seen in Boyer-Lindquist coordinates [**MTW**], [**W**].

What is more important for us is that these solutions are actually *families* of solutions of the Einstein equations. In fact by thinking of a fixed asymptotically flat coordinate system near infinity in the spacetime, by varying the total mass and angular momentum, and considering families of spacelike slices of these metrics, we get a ten-parameter family of asymptotically flat solutions of the vacuum constraint equations near infinity in $\mathbb{R}^3$. The purpose of considering this family is to control the energy-momentum $(E, \mathbf{P})$ and the angular momentum $\mathbf{J}$ at (spacelike) infinity, as well as a quantity $\mathbf{C}$ related to the center-of-mass. In an asymptotically flat chart these quantities are defined as limits of integrals over Euclidean spheres with surface measure $d\sigma_g$ and outer normal ν taken with respect to g

$$E = \frac{1}{16\pi} \lim_{R\to\infty} \oint\limits_{|x|=R} \sum_{i,j} (g_{ij,i} - g_{ii,j})\, \nu^j d\sigma_g, \tag{1}$$

$$P_i = \frac{1}{8\pi} \lim_{R\to\infty} \oint\limits_{|x|=R} \sum_{j} \pi_{ij} \nu^j d\sigma_g, \tag{2}$$

$$J_i = \frac{1}{8\pi} \lim_{R\to\infty} \oint\limits_{|x|=R} \sum_{j,k} \pi_{jk} Y_i^j \nu^k d\sigma_g, \tag{3}$$

$$\begin{aligned} C^k = & \lim_{R\to\infty} \oint\limits_{|x|=R} \sum_{i,j} x^k \,(g_{ij,i} - g_{ii,j})\, \nu^j d\sigma_g \\ & - \lim_{R\to\infty} \oint\limits_{|x|=R} \sum_{i} \left(g_{ik}\nu^i - g_{ii}\nu^k\right) d\sigma_g. \end{aligned} \tag{4}$$

In the case when g is AF, conformally flat near infinity and has zero scalar curvature, the quantity $\mathbf{C}$ is proportional to $m\mathbf{c_0}$, where m is the mass and $\mathbf{c_0}$ is the coordinate translation which makes the $|x|^{-2}$-terms in the expansion of the conformal factor vanish [**C**]. We note that it may be useful to shift the quantity $\mathbf{C}$ (for example an initial shift of coordinates will do this), and also we note that the vector fields Y_i are the basic rotation fields in $\mathbb{R}^3$, for example

$$Y_1 = x^2 \frac{\partial}{\partial x^3} - x^3 \frac{\partial}{\partial x^2}.$$

The integrand for the angular momentum is essentially the cross product of the position vector with the momentum tensor, in analogy with the angular momentum of classical mechanics.

We can prescribe conditions under which the above asymptotic integrals converge for *solutions* of the vacuum constraints. These conditions are asymptotic even/odd conditions on the metric tensor and second fundamental form of the initial data; we call this condition (AC). The conditions are

$$\begin{aligned} g_{ij}(x) &= \delta_{ij} + O(|x|^{-1}) \quad K_{ij}(x) = O(|x|^{-2}), \\ g_{ij}(x) - g_{ij}(-x) &= O(|x|^{-2}) \quad K_{ij}(x) + K_{ij}(-x) = O(|x|^{-3}), \\ g_{ij,k}(x) + g_{ij,k}(-x) &= O(|x|^{-3}) \quad K_{ij,k}(x) - K_{ij,k}(-x) = O(|x|^{-4}). \end{aligned} \tag{5}$$

We require analogous conditions on successive derivatives as needed, and these will be implied in the "O"-notation for the (AF) and (AC) conditions. It is also worthwhile to remark that under these asymptotic conditions on solutions of the constraints, the corresponding boundary integrals above can also be computed with respect to the Euclidean metric, and the limiting values are the same, a fact we may use without further comment.

In Section 3 of this paper we show that the vacuum initial data sets which satisfy (AC) are dense in a suitable weighted Sobolev space $(g_{ij} - \delta_{ij}) \in W^{2,p}_{-\delta}$ and $\pi_{ij} \in W^{1,p}_{-1-\delta}$ for $p > 3/2$, $\delta \in (1/2, 1)$ in the set of all vacuum initial data which satisfy the standard decay assumptions (but not the asymptotic symmetry). This space is strong enough that the total energy and linear momentum are continuous on the space. (Of course, the angular momentum is not continuous, nor does it appear to be well-defined on this more general space.) This theorem is an analogue for the full constraint equations of the theorem of Schoen-Yau [**SY**] in the time-symmetric case ($K = 0$). It is however much more subtle to prove because the general constraint operator is not as well understood as the time-symmetric operator. We show that it is possible to achieve a very special type of asymptotic behavior which in particular satisfies (AC). These special asymptotic conditions require that outside a compact set there exist a positive function u and a vector field X so that $g_{ij} = u^4\delta_{ij}$ and $\pi_{ij} = u^2(L_X\delta - \operatorname{div}_\delta(X)\delta)_{ij}$ where $L_X\delta$ denotes the Lie derivative of the Euclidean metric δ with respect to X. It is not difficult to see that such data automatically satisfies (AC). An important feature of these asymptotic conditions is that the total energy and momenta (both linear and angular) may be read off of the asymptotics of (u, X), and the behavior of these conserved quantities directly affects the asymptotic geometry. This type of asymptotic behavior will be discussed in detail by the second author in future work. The proof of the density of solutions involves taking an arbitrary asymptotically flat solution (g, π) of the vacuum constraint equations and patching g to the Euclidean metric and π to 0 in a large annulus. It is then necessary to reimpose the constraints by solving a partial differential equation for (u, X). The cut-off data may be viewed as an approximate solution, and one may attempt to use the inverse function theorem to correct it to a solution. The problem which arises is that it is not clear whether the corresponding linearization is an isomorphism in the natural spaces. This problem is overcome by allowing the extra flexibility of addition of a suitably chosen family of compactly supported deformations (of g, π). In order to make this work, we need to use the fact that the constraint operator (at an arbitrary asymptotically flat initial data set) has surjective linearization in these weighted Sobolev spaces. This result is proven for maximal data by Choquet-Bruhat, Fischer, and Marsden in [**CFM**], and the general case is due to Beig and Ó Murchadha [**BO**].

The second density theorem (Theorem 5) will follow from a gluing construction (Theorem 4) coupled with the first density theorem. We now outline the basic approach to this gluing method, the proof of which occupies Sections 4 and 5. Given asymptotically flat initial data on M satisfying (AC) in an appropriate chart at infinity in a given end, we take a sufficiently large radius R and within the annulus from R to $2R$, we smoothly patch the given metric and second fundamental form to the metric and second fundamental form coming from a slice in Kerr, or from another suitable family satisfying (AC) near infinity (cf. Section 5). This will produce an approximate solution of the vacuum constraints. The approximate solution is altered with a smooth perturbation compactly supported within the closed annular gluing region to data $(\bar{g}, \bar{\pi})$, whose constraint function $\Phi(\bar{g}, \bar{\pi})$ lies in a finite-dimensional vector space of dimension ten; of course we want $\Phi(\bar{g}, \bar{\pi})$ to be zero. (Note that outside the annulus the constraint function vanishes by design.) As in [**C**], the key to the proof is exploiting the overdetermined-ellipticity of the adjoint of the linearization of Φ. We then show that there is a choice of data to glue on so that the resulting perturbed metric and second fundamental form are solutions to the constraint equations. The ten-dimensional obstruction space is countered by the ten degrees of freedom afforded by $(E, \mathbf{P}, \mathbf{J}, \mathbf{C})$; in fact the method proves that suitable models near infinity are precisely those for which we can effectively attain values of $(E, \mathbf{P}, \mathbf{J}, \mathbf{C})$ by moving within the family. The conclusion of the proof is the observation that the map from the parameter space to the obstruction space, a map between two ten-dimensional Euclidean spaces, has non-zero degree; this map is more complicated than before (where only four parameters were needed), but recognizing the (AC) condition affords some economy in computing this map as compared to [**C**], where an expansion of the conformal factor is used.

We remark that the results of this paper are easily seen to allow M to have multiple asymptotically flat ends. For example, the proof of Theorem 4 is a construction local to any given end. For convenience of notation we may write the proofs under the assumption that M has only one end.

The results of this paper (with the exception of Section 3) were announced in the spring of 2000, and have been widely communicated to the mathematical GR community. In the meantime, P.T. Chruściel and E. Delay [**CD**] have obtained a version of these results as well. Their paper employs our basic techniques (and those of the first author in [**C**]) to give a number of interesting applications; in addition their paper gives an elegant and explicit description of the 10-parameter Kerr family of initial data which we do not do here.

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2. Preliminaries

2.1. Basic Notation. Let $\Omega \subset\subset M$ denote a compactly contained domain (i.e., $\overline{\Omega}$ is compact) in a smooth three-manifold M. Unless noted, we assume the boundary is smooth. We list here some notation, and we define some function spaces which we will find useful.

- $\mathrm{Ric}\,(g) = R_{ij}$ and $R(g) = g^{ij}R_{ij}$ denote the Ricci and scalar curvatures, respectively, of a Riemannian metric g on M. We use the Einstein summation convention throughout, as well as the convention of using a semicolon to denote covariant differentiation and a comma to denote partial differentiation, and our convention for the Laplacian is $\Delta_g f = \frac{1}{\sqrt{|g|}}\,\partial_i(g^{ij}\sqrt{|g|}\,\partial_j f)$.
- We let $d\mu_g$ denote the volume measure induced by g, $d\sigma_g$ the induced surface measure on submanifolds, dx the Lebesgue measure on Euclidean space, and $d\xi$ the Euclidean surface measure.
- We denote by H^k the Hilbert space of tensor fields which are square-integrable along with the first k weak covariant derivatives, with the standard H^k-inner product induced by the metric g. It will be clear in context which type of fields are being discussed. H^k_{loc} denotes spaces of tensors which are in H^k on each compact subset. We may abbreviate products with mixed regularity by using superscripts, like $H^{2,1} = H^2 \times H^1$.
Similarly we have the Hölder spaces $C^{k,\alpha}$.
- We may wish to use symbols to distinguish various types of tensors. For example, let $\mathcal{M}^k$ ($k > \frac{3}{2}$) denote the open subset of H^k of Riemannian metrics, and $\mathcal{M}^{k,\alpha}$ denotes the open subset of metrics

in $C^{k,\alpha}$. $\mathcal{S}_{(2,0)}$ denotes the space of symmetric (2,0)-tensor fields, and $\mathcal{X}$ denotes the space of vector fields. We use superscripts as above to denote the desired Sobolev or Hölder regularity of the fields.

- Let ρ be a smooth positive function on Ω. Define $L^2_\rho(\Omega)$ to be the set of locally $L^2(d\mu_g)$ functions f such that $f\rho^{1/2} \in L^2(\Omega, d\mu_g)$. The pairing
$$\langle f_1, f_2\rangle_{L^2_\rho(\Omega)} = \langle f_1\rho^{1/2}, f_2\rho^{1/2}\rangle_{L^2(\Omega,d\mu_g)}$$
makes $L^2_\rho(\Omega)$ a Hilbert space. It is clear how to extend this definition to higher order tensors.
- Let $H^k_\rho(\Omega)$ be the Hilbert space of tensor fields in $L^2_\rho(\Omega)$ along with the first k covariant derivatives, the inner product defined by incorporating the $L^2_\rho(\Omega)$-pairings on the covariant derivatives of order $0, \ldots, k$.
- We now want to define the weighted Hölder spaces $C^{k,\alpha}_{\rho^{-1}}(\Omega)$ $(0 < \alpha < 1)$. We will consider ρ which near $\partial\Omega$ decays as a power of or exponentially in the distance to the boundary. The weighted Hölder space is defined as the subspace of $C^{k,\alpha}(\overline{\Omega})$ comprised of functions f for which the norm
$$\|f\|_{k,\alpha,\rho^{-1}} := \|f\rho^{-\frac{1}{2}}\|_{k,\alpha}$$
is finite; this is a Banach space. (Unless noted otherwise, norms will be taken over Ω.)
- For an AF (M, g), we will use weighted Sobolev spaces $W^{k,p}_{-\delta}(M, g)$ to capture asymptotics of functions and tensors near infinity. The weighted norm convention we are using is that the $W^{k,p}_{-\delta}$ norm is given by
$$\|f\|_{k,p,-\delta} = \sum_{0\le|\alpha|\le k} \left(\int_M (|D^\alpha f|\rho^{\delta+|\alpha|})^p \rho^{-3} d\mu_g\right)^{1/p}$$
where in this context ρ is a function which equals $|x|$ near infinity (in an AF chart) and α is a multi-index.

The following lemma concerning the density of $H^k(\Omega)$ in $H^k_\rho(\Omega)$ will be useful.

Lemma 2.1. *Assume that ρ is bounded from above. For $k \ge 1$, the subspace $H^k(\Omega)$ (and hence $C^\infty(\overline{\Omega})$) is dense in $H^k_\rho(\Omega)$.*

Proof. For small $\tau > 0$ let $\widetilde{\Omega}_\tau$ be a slightly expanded domain compactly containing Ω. Let $F_\tau : \widetilde{\Omega}_\tau \to \Omega$ be a diffeomorphism which is C^k-close to identity map. Now suppose that $f \in H^k_\rho(\Omega)$ and let $f_\tau = f \circ F_\tau$. Note that the restriction of f_τ to Ω is in $H^k(\Omega)$. We will show that $f_\tau \to f$ in $H^k_\rho(\Omega)$. First note that by the chain rule we have $|D^\alpha f_\tau| \leq c \sum_{|\beta|\leq|\alpha|} |D^\beta f| \circ F_\tau$ for $|\alpha| \leq k$, where c is independent of τ. Now, the dominated convergence theorem implies that $\lim_{\delta\to 0^+} \int_{\Omega\setminus\Omega_\delta} |D^\alpha f|^2 \rho d\mu_g = 0$ for $|\alpha| \leq k$, where Ω_δ denotes the subset of Ω consisting of points at least a distance δ from the boundary. It follows that for $\tau > 0$ we also have $\lim_{\delta\to 0^+} \int_{\Omega\setminus\Omega_\delta} |D^\alpha f_\tau|^2 \rho d\mu_g = 0$, uniformly in τ small. Thus, given $\epsilon > 0$ we may choose δ so small that $(\int_{\Omega\setminus\Omega_\delta} |D^\alpha f_\tau - D^\alpha f|^2 \rho d\mu_g)^{1/2} < \epsilon/2$, and then choose τ so small that $(\int_{\Omega_\delta} |D^\alpha f_\tau - D^\alpha f|^2 \rho d\mu_g)^{1/2} < \epsilon/2$. Summing these completes the proof. q.e.d.

2.2. Linearization of the Constraint Map. We gather here a few facts we will use in what follows [**FM1**], [**FM2**].

Lemma 2.2. *The scalar curvature map is a smooth Banach space map, as a map $R : \mathcal{M}^{l+2}(\Omega) \to H^l(\Omega)$ ($l > \frac{1}{2}$), or $R : \mathcal{M}^{k+2,\alpha}(\overline{\Omega}) \to C^{k,\alpha}(\overline{\Omega})$ ($k \geq 0$). The linearization L_g of the scalar curvature operator is given by*

$$L_g(h) = -\Delta_g(Tr_g(h)) + \operatorname{div}_g(\operatorname{div}_g(h)) - h \cdot \operatorname{Ric}(g)$$

in either of the above spaces. The formal $L^2(d\mu_g)$-adjoint L_g^* of L_g is given by

$$L_g^*(f) = -(\Delta_g f)g + \operatorname{Hess}_g(f) - f \operatorname{Ric}(g). \tag{6}$$

Lemma 2.3. *The constraint map Φ is smooth Banach space map, as a map $\Phi : \mathcal{M}^{l+2}(\Omega) \times \mathcal{S}^{l+2}_{(2,0)}(\Omega) \to H^l(\Omega) \times \mathcal{X}^{l+1}(\Omega)$ ($l > \frac{1}{2}$), or $\Phi : \mathcal{M}^{k+2,\alpha}(\overline{\Omega}) \times \mathcal{S}^{k+2,\alpha}_{(2,0)}(\overline{\Omega}) \to C^{k,\alpha}(\overline{\Omega}) \times \mathcal{X}^{k+1,\alpha}(\overline{\Omega})$ ($k \geq 0$). The formal $L^2(d\mu_g)$-adjoint $D\Phi^*_{(g,\pi)}$ of the linearization $D\Phi_{(g,\pi)}$ is given by $D\Phi^*_{(g,\pi)}(f, X) = D\mathcal{H}^*_{(g,\pi)}(f) + D\operatorname{div}^*_{(g,\pi)}(X)$ where*

$$D\mathcal{H}^*_{(g,\pi)}(f) = \Big((L_g^* f)_{ij} + \big((Tr_g\pi)\pi_{ij} - 2\pi_{ik}\pi^k_j\big)f, \\ \big((Tr_g\pi)g^{ij} - 2\pi^{ij}\big)f\Big),$$

$$D\operatorname{div}^*_{(g,\pi)}(X) = \frac{1}{2}\Big((L_X\pi)_{ij} + (X^k_{;k})\pi_{ij} - (X_i\pi^k_{j;k} + \pi^k_{i;k}X_j) \\ - X_{k;m}\pi^{km}g_{ij} - X_k\pi^{km}_{;m}g_{ij}, \ \ -(L_X g)^{ij}\Big).$$

We point out here that the key terms we will use are the $L_g^* f$-term in the first component above, and the Lie derivative term $L_X g$ in the second component. The formula is well-known and straightforward to derive, although we note that in [**FM2**], for example, the negative of the divergence operator is used, so some signs differ.

We note that in some sources, like [**CFM**], the tensor K is treated as a $(1,1)$-tensor, and if we let the constraint operator in this form be

$\Psi(g,K) = (R(g) - |K|^2 + H^2, \operatorname{div}_g(K) - dH)$, then the adjoint of $D\Psi$ is somewhat easier to compute and has the form

$$\begin{aligned} D\Psi^*(\xi, Z) = \big(&L_g^*\xi - 1/2((K^{pq}Z_q)_{;p})g_{ij} + 1/2(K_{ij}Z^p)_{;p}, \\ &- 1/2(L_Z g)_{ij} + (\operatorname{div} Z)g_{ij} - 2\xi K_{ij} + 2\xi(Tr_g K)g_{ij}\big). \end{aligned} \tag{7}$$

The leading terms are of course the same as in the other formulation.

We will also use the following fact, which is proved in an analogous fashion to the lemmas above, using Sobolev embedding.

Lemma 2.4. *Let $p > 3/2$ and $\delta \in (0,1)$. For $(g_{ij} - \delta_{ij}, \pi_{ij}) \in W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$, the map $(h,k) \mapsto \Phi(g+h, \pi+k)$ is continuously differentiable from a neighborhood of $(0,0)$ in $W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$ to $W^{0,p}_{-2-\delta}$.*

We now indicate the source of the ten-dimensional obstruction space we mentioned in the previous section.

Lemma 2.5. *Let Ω be an open domain in $\mathbb{R}^3$ with flat initial data $(\delta, 0)$. Then on Ω the kernel K of the operator $D\Phi^*_{(\delta,0)}$ is the direct sum of the span K_0 of the functions 1, x^i $(i = 1,2,3)$ and the span K_1 of the vector fields $X_i = \frac{\partial}{\partial x^i}$ $(i = 1,2,3)$ and $Y_k = x^i \frac{\partial}{\partial x^j} - x^j \frac{\partial}{\partial x^i}$ $(i < j$, $k \neq i,j)$.*

Proof. By Lemma 2.3, $D\Phi^*_{(\delta,0)}(f,X) = (L^*_\delta f, -\frac{1}{2}(L_X\delta)^\sharp)$. Moreover, by Eq.(2.2), $L^*_\delta f = 0$ implies $\operatorname{Hess}_\delta f = 0$, i.e., $f \in K_0$. That $L_X\delta = 0$ means X is a Killing field of the flat metric on $\mathbb{R}^3$, so $X \in K_1$. q.e.d.

We note that the kernel of $D\Phi^*_{(g,\pi)}$ has a natural interpretation. We first recall that it is straightforward to show [**C**] that a nontrivial element f in the kernel of L^*_g yields a warped product metric $-f^2dt^2 + g$ that is Einstein, and if $R(g) = 0$ then this product metric is a solution to the Einstein vacuum equations which is manifestly static. Moncrief [**M**] showed that a nontrivial element (f,X) in the kernel of $D\Phi^*_{(g,\pi)}$ at a solution of the constraints corresponds to a Killing field (symmetry) in the resulting vacuum spacetime.

3. Constructing Solutions with Good Asymptotics

In this section we show that any solution of the vacuum constraint equations with $g_{ij} = \delta_{ij} + O(1/r)$ and $\pi_{ij} = O(1/r^2)$ may be perturbed by an arbitrarily small amount on any compact set to a new solution satisfying the condition (AC). We also show that the ADM energy-momentum vector is stable under this approximation in that it is perturbed by an arbitrarily small amount. This construction shows that the gluing theorem (Theorem 4) of the paper applies to a very large class of solutions of the vacuum constraint equations which are dense in a suitable sense.

The construction we are going to make may be viewed as a generalization of the deformation result of [**SY**] from the time-symmetric case to the case of arbitrary data. We begin with any asymptotically flat data (g, π) satisfying the vacuum constraint equations and such that $g_{ij} = \delta_{ij} + O(1/r)$ and $\pi_{ij} = O(1/r^2)$. We are going to show how to approximate this data by new data $(\bar{g}, \bar{\pi})$ satisfying the vacuum constraint equations and such that outside a compact set we have

$$\bar{g}_{ij} = u^4 \delta_{ij},\ \bar{\pi}_{ij} = u^2 (\mathcal{L}_\delta X)_{ij} \tag{8}$$

where u tends to 1 at infinity and X is a vector field tending to 0 at infinity, and where $\mathcal{L}_g$ is the operator related to the Lie derivative $L_X g$ by

$$\mathcal{L}_g X = L_X g - \operatorname{div}_g(X) g.$$

If such asymptotics can be achieved, then the constraint equations near infinity become the equations (computed with respect to δ)

$$8\Delta u = u\big(-|\mathcal{L}X|^2 + 1/2(Tr(\mathcal{L}X))^2\big),$$
$$\Delta X^i + 4u^{-1} u_j (\mathcal{L}X)_i^j - 2u^{-1} u_i Tr(\mathcal{L}X) = 0$$

where the second equation is written with respect to a Euclidean basis. Standard asymptotics (see [**B**]) then imply that

$$u(x) = 1 + a/r + O(1/r^2), \quad X^i = b^i/r + O(1/r^2)$$

for constants a, b^i. These asymptotics clearly imply (AC).

In the following theorem we will generalize the above discussion to allow solutions which are in the weighted Sobolev spaces $W^{k,p}_{-\delta}$ defined above.

Theorem 1. *Let $(g_{ij} - \delta_{ij}, \pi_{ij}) \in W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$ be a vacuum initial data set, where $\delta \in (1/2, 1)$ and $p > 3/2$. Given any $\epsilon > 0$, there exists a vacuum initial data set $(\bar{g}, \bar{\pi})$ satisfying (8) which is within ϵ of (g, π) in the $W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$ norm. Moreover, the mass and linear momentum of $(\bar{g}, \bar{\pi})$ are within ϵ of those of (g, π).*

Proof. In order to show that (8) can be achieved, we begin by modifying (g, π) to $(\hat{g}, \hat{\pi})$ in the annular region from R to $2R$ so that $\hat{g} = \delta$ and $\hat{\pi} = 0$ outside the ball B_{2R}. We then attempt to reimpose the constraint equations by constructing a solution of the form $\bar{g} = u^4 \hat{g}$ and $\bar{\pi} = u^2(\hat{\pi} + \mathcal{L}X)$, where the operator $\mathcal{L} = \mathcal{L}_{\hat{g}}$ is computed with respect to the metric $\hat{g}$. We hope to find a solution u which is near 1 and X which is near 0. The constraint equations then become the system

$$\begin{aligned}\bar{\mu} := \mathcal{H}(\bar{g}, \bar{\pi}) = u^{-5}[&-8\Delta_{\hat{g}} u + u(R(\hat{g}) - |\hat{\pi} + \mathcal{L}X|^2_{\hat{g}} \\ &+ 1/2(Tr_{\hat{g}}(\hat{\pi} + \mathcal{L}X))^2)] = 0,\\ (\operatorname{div}_{\bar{g}}(\bar{\pi}))_i = u^{-2}[&(\operatorname{div}_{\hat{g}}(\hat{\pi} + \mathcal{L}X))_i + 4u^{-1} u_j (\hat{\pi} + \mathcal{L}X)_i^j \\ &- 2u^{-1} u_i Tr_{\hat{g}}(\hat{\pi} + \mathcal{L}X)] = 0.\end{aligned}$$

We consider the map $T(u,X) = \Phi(\bar{g},\bar{\pi}) = (\bar{\mu}, \operatorname{div}_{\bar{g}}(\bar{\pi}))$ and observe that the linearization at $(1,0)$ is given by

$$DT(\eta, Y) = \Big(-8\Delta\eta - 4\hat{\mu}\eta - 4\hat{\pi}^{ij}Y_{i;j} + (\operatorname{div} Y)Tr\hat{\pi},$$
$$\operatorname{div}(\mathcal{L}Y)_i + 4\eta_j\hat{\pi}_i^j - 2\eta_i Tr\hat{\pi} - 2\eta \operatorname{div}\hat{\pi}\Big)$$

where each term is computed with respect to $\hat{g}$. In order to solve the equation $T(u,X) = 0$, we wish to use the inverse function theorem. Thus we would need to check that $DT = DT_{(\hat{g},\hat{\pi})}$ is an isomorphism between appropriate spaces with bounded inverse (independent of R for large enough R). To be precise, we consider DT as an operator from $W^{2,p}_{-\delta} \times W^{2,p}_{-\delta}$ to $W^{0,p}_{-2-\delta}$. It is well known that DT is a Fredholm operator of index 0 for $p > 1$ and $\delta \in (0,1)$ (see [**B**]). Thus in order to check that DT is an isomorphism, it would suffice to check that the cokernel is trivial; i.e., that DT is onto. This seems to be difficult to check in general, so we use a less direct approach. We consider enlarging the domain and we look for solutions of the modified problem

$$\Phi(u^4\hat{g} + h, u^2(\hat{\pi} + \mathcal{L}X) + k) = 0$$

where h and k are suitably small symmetric $(0,2)$-tensors with *compact support*. If we can solve this problem, the desired asymptotics follow as before.

We will need to use the fact that the operator $D\Phi = D\Phi_{(\hat{g},\hat{\pi})}$ is surjective as an operator from $W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$ to $W^{0,p}_{-2-\delta}$ for $\delta \in (0,1)$. This result was proven by Choquet-Bruhat, Fischer, and Marsden in [**CFM**] in the maximal case, and by Beig and Ó Murchadha [**BO**] in the general case. We give a direct proof in the next proposition which appears to work under weaker asymptotic assumptions than those of [**BO**], although the same basic idea is employed.

Proposition 3.1. *The operator $D\Phi_{(\hat{g},\hat{\pi})}$ is surjective from the domain $W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$ onto $W^{0,p}_{-2-\delta}$ for $p > 1$ and $\delta \in (0,1)$.*

Proof. We will check the surjectivity for the initial data (g,π), and the result for $(\hat{g},\hat{\pi})$ will follow by a perturbation argument. We first note the range of $D\Phi$ contains that of DT (a closed finite codimension subspace), and hence is of finite codimension in $W^{0,p}_{-2-\delta}$. Therefore $D\Phi$ has closed range. If there is an element (ξ, Z) in the dual space $W^{0,q}_{-1+\delta}$ which annihilates the range of $D\Phi$ (which equals the range of $D\Psi$), we have that (ξ, Z) lies in the kernel of $D\Psi^*$ and hence by (7) satisfies the equations:

$$\xi_{;ij} - (\Delta\xi)g_{ij} - \xi R_{ij} - 1/2((K^{pq}Z_q)_{;p})g_{ij} + 1/2(K_{ij}Z^p)_{;p} = 0,$$
$$-1/2(Z_{i;j} + Z_{j:i}) + (\operatorname{div} Z)g_{ij} - 2\xi K_{ij} + 2\xi(TrK)g_{ij} = 0.$$

We may take the trace and rewrite the equations eliminating the terms involving $\Delta\xi$ and $\operatorname{div} Z$. We then have

$$
\begin{aligned}
(9)\qquad & \xi_{;ij} - \xi R_{ij} + 1/2(K_{ij}Z^p)_{;p} \\
& \quad + [1/2R\xi + 1/4(K^{pq}Z_q)_{;p} - 1/4((TrK)Z^p)_{;p}]g_{ij} = 0, \\
& 1/2(Z_{i;j} + Z_{j:i}) + 2\xi K_{ij} = 0.
\end{aligned}
$$

Note also that taking the trace of the first equation and the divergence of the second, we get a system of linear equations of the form $\Delta(\xi, Z) = B(x)(\nabla\xi, \nabla Z) + C(x)(\xi, Z)$, where B, C are coefficient matrices. The following estimates are true for any weight $\tau > 0$ and for any large radius R (where $(L_Z g)_{ij} = Z_{i;j} + Z_{j;i}$ is the Lie derivative):

$$
\begin{aligned}
(10)\qquad & \int_{M\setminus B_R} |\xi\rho^\tau|^2\rho^{-3}d\mu_g \le C\int_{M\setminus B_R}(|\nabla\nabla\xi|\rho^{2+\tau})^2\rho^{-3}d\mu_g, \\
& \int_{M\setminus B_R}(|Z|\rho^\tau)^2\rho^{-3}d\mu_g \le C\int_{M\setminus B_R}(|L_Z g|\rho^{1+\tau})^2\rho^{-3}d\mu_g.
\end{aligned}
$$

A proof of these inequalities will be given below. Using these inequalities, we can show inductively that (ξ, Z) must vanish to infinite order at infinity; i.e., $|(\xi, Z)| \le C_N\rho^{-N}$ for any integer $N > 1$. To see this, we know from simple initial asymptotics for the equations that ξ and Z are of order ρ^{-1}. Putting this into the equations we have that $\nabla\nabla\xi$ is of order ρ^{-4} and $L_Z g$ is of order ρ^{-3}. Thus we may choose $\tau < 2$ and conclude that (ξ, Z) are of order at most $\rho^{-\tau}$ for any $\tau < 2$. Note that the pointwise bounds follow from the corresponding weighted L^2 bounds by applying standard elliptic estimates (mean value-type inequalities) on balls of the form $B_{|x|/2}(x)$. Putting this information back into the equations, we find that $\nabla\nabla\xi$ is of order $\rho^{-\tau-3}$ and $L_Z g$ is of order $\rho^{-\tau-2}$. Thus we may choose $\tau < 3$ and improve the decay on (ξ, Z). We thus conclude inductively that (ξ, Z) vanishes to infinite order at infinity.

To show that (ξ, Z) vanishes identically, we use a standard unique continuation result (see Kazdan [**K**] for a version written in the most convenient form). To see this, we consider doing an inversion (Kelvin transform). That is, we let $x = (x^1, x^2, x^3)$ denote asymptotically Euclidean coordinates and we introduce $y = |x|^{-2}x$ where $|x|$ is the Euclidean norm of x. Now observe that the metric $\bar{g} = |x|^{-4}g$ has scalar curvature $\bar{R} = |x|^5(-8\Delta(|x|^{-1}) + |x|^{-1}R)$ and this is of order $|x|$ at infinity. We then observe that the metric $\bar{g}$ expressed in the y coordinates has Lipschitz components near $y = 0$, and that $\bar{g}_{ij}(0) = \delta_{ij}$. The conformal transformation for the Laplace operators is then $\Delta u - 1/8Ru = |y|^5(\bar{\Delta}(|y|^{-1}u) - 1/8\bar{R}(|y|^{-1}u))$. It follows that the quantities $(\bar{\xi}, \bar{Z}) = (|y|^{-1}\xi, |y|^{-1}Z)$ satisfy a linear system of the form

$$\bar{\Delta}(\bar{\xi}, \bar{Z}) = \bar{B}(y)(\bar{\nabla}\bar{\xi}, \bar{\nabla}\bar{Z}) + \bar{C}(y)(\bar{\xi}, \bar{Z})$$

with $\bar{B}$ is bounded near $y = 0$ and $\bar{C}$ is bounded by a constant times $|y|^{-1}$. The unique continuation theorem then implies that (ξ, Z) vanishes identically.

Proof of (10). We choose a smooth cutoff function ζ which takes values between 0 and 1 with $\zeta(x) = 0$ for $|x| \le R$ and $\zeta(x) = 1$ for $|x| \ge 2R$. The following inequalities for the Euclidean metric δ on $\mathbb{R}^3$, and for functions f and vector fields X in the appropriate weighted spaces, are standard and can be proved using integration by parts:

$$\begin{aligned} \int_{\mathbb{R}^3} |f\rho^\tau|^2 \rho^{-3} dx &\le C \int_{\mathbb{R}^3} (|\nabla f| \rho^{1+\tau})^2 \rho^{-3} dx, \\ \int_{\mathbb{R}^3} (|X| \rho^\tau)^2 \rho^{-3} dx &\le C \int_{\mathbb{R}^3} (|L_X \delta| \rho^{1+\tau})^2 \rho^{-3} dx. \end{aligned}$$

Thus we may take $f = \zeta\xi$ and $X = \zeta Z$ to obtain

$$\begin{aligned} \int_M |\zeta\xi\rho^\tau|^2 \rho^{-3} dx &\le C \int_M (|\nabla(\zeta\xi)| \rho^{1+\tau})^2 \rho^{-3} dx, \\ \int_M (|\zeta Z| \rho^\tau)^2 \rho^{-3} dx &\le C \int_M (|L_{\zeta Z} \delta| \rho^{1+\tau})^2 \rho^{-3} dx \end{aligned}$$

where all quantities are taken with respect to the Euclidean metric on the end $M \setminus B_R$. Using the choice of ζ and easy manipulations, these inequalities clearly imply

$$\begin{aligned} &\int_{M \setminus B_R} |\xi\rho^\tau|^2 \rho^{-3} dx \\ &\le C \int_{M \setminus B_R} (|\nabla\xi| \rho^{1+\tau})^2 \rho^{-3} dx + C R^{2\tau-3} \int_{B_{2R} \setminus B_R} \xi^2 dx, \\ &\int_{M \setminus B_R} (|Z| \rho^\tau)^2 \rho^{-3} dx \\ &\le C \int_{M \setminus B_R} (|L_Z \delta| \rho^{1+\tau})^2 \rho^{-3} dx + C R^{2\tau-3} \int_{B_{2R} \setminus B_R} |Z|^2 dx. \end{aligned} \tag{11}$$

We may now prove (10) for the Euclidean metric by contradiction. For example, to prove the second inequality, suppose we have a sequence Z_α with

$$\int_{M \setminus B_R} (|Z_\alpha| \rho^\tau)^2 \rho^{-3} dx = 1$$

but

$$\int_{M \setminus B_R} (|L_{Z_\alpha} \delta| \rho^{1+\tau})^2 \rho^{-3} dx \to 0.$$

We may assume that the Z_α converge H^1-weakly and in L^2_{loc} to a limit Z. By the inequality (11) above we see that $Z \ne 0$, and we must have $L_Z \delta = 0$. Therefore Z is a nonzero Killing vector field for δ with $\int_{M \setminus B_R} (|Z| \rho^\tau)^2 \rho^{-3} dx < \infty$. This is a contradiction which proves the second inequality of (10) for the Euclidean metric. By a similar argument one proves

$$\int_{M\setminus B_R} |\xi\rho^\tau|^2\rho^{-3}dx \leq C\int_{M\setminus B_R}(|\nabla\xi|\rho^{1+\tau})^2\rho^{-3}dx,$$

and the first inequality of (10) follows from applying this with ξ replaced by $\nabla\xi$ and τ replaced by $\tau+1$. Using the fact that g approaches the Euclidean metric on approach to infinity, the inequalities (10) follow. This completes the proof of the surjectivity of $D\Phi$. q.e.d.

We are now in a position to complete the proof of Theorem 1. It is well known (Lemma 2.4) that $(h,k)\mapsto\Phi(g+h,\pi+k)$ is a continuously differentiable map from a neighborhood of $(0,0)$ in $W^{2,p}_{-\delta}\times W^{1,p}_{-1-\delta}$ to $W^{0,p}_{-2-\delta}$ for $p>3/2$ and $\delta\in(0,1)$. We further observe that the truncated data $(\hat{g},\hat{\pi})$ are arbitrarily close (for R large) to (g,π) in the $W^{2,p}_{-\delta}\times W^{1,p}_{-1-\delta}$ norm. Thus the linearization $D\Phi$ is also surjective at $(\hat{g},\hat{\pi})$. We may choose a basis $V_1,\ldots,V_N$ for the cokernel of DT, and choose $(0,2)$-tensors $(h_1,k_1),\ldots,(h_N,k_N)$ in $W^{2,p}_{-\delta}\times W^{1,p}_{-1-\delta}$ so that $D\Phi(h_s,k_s)=V_s$ for $s=1,\ldots,N$. We now perturb the (h_s,k_s) to make them of compact support. The corresponding images under $D\Phi$ are close to the V_s, and hence they still span a complementing subspace for the closed subspace $Im(DT)$. We let W_2 be the linear span of $(h_1,k_1),\ldots,(h_N,k_N)$. We note that the null space $U\subseteq W^{2,p}_{-\delta}\times W^{2,p}_{-\delta}$ of DT is N-dimensional since the index of DT is 0. We let W_1 be a closed complementing subspace to U, and we let $W=W_1\times W_2$, so that W is a Banach space. We then define the map $\overline{T}$ on an open neighborhood of $((1,0),(0,0))\in((1,0),(0,0))+W$ by setting $\overline{T}((u,X),(h,k))=\Phi(u^4\hat{g}+h,u^2(\hat{\pi}+\mathcal{L}X)+k)$. We see by construction that $D\overline{T}$ is an isomorphism at $((1,0),(0,0))$, so we may apply the standard inverse function theorem to assert that $\overline{T}$ is an isomorphism from a fixed (independent of R) neighborhood of $((1,0),(0,0))$ and covers a fixed neighborhood of $\Phi(\hat{g},\hat{\pi})$. It then follows that this image contains $(0,0)$, so we may find u near 1, X near 0, (h,k) near $(0,0)$ in the appropriate spaces so that $\Phi(\bar{g},\bar{\pi})=0$, where $\bar{g}=u^4\hat{g}+h$ and $\bar{\pi}=u^2(\hat{\pi}+\mathcal{L}X)+k$.

We now show that the ADM energy and linear momentum are perturbed by a small amount under this perturbation of (g,π). If E,P_i denote the total energy and linear momentum of a solution (g,π) of the vacuum constraint equations, we need only show that E,P_i are continuous with respect to the $W^{2,p}_{-\delta}\times W^{1,p}_{-1-\delta}$ norm for $\delta>1/2$. For example, to see this for the P_i, observe that by the divergence theorem we have for $R_1>R$

$$\oint\limits_{|\mathbf{x}|=R_1}\pi_{ij}\nu^j d\sigma_g-\oint\limits_{|\mathbf{x}|=R}\pi_{ij}\nu^j d\sigma_g=\int_{B_{R_1}\setminus B_R}\pi^j{}_{i;j}d\mu_g$$

where ${\pi^j}_{i;j}$ represents the divergence of the *vector field* ${\pi^j}_i\partial/\partial x^j$. Using the constraint equation, we see that the volume integrand is a sum of Christoffel symbol times components of π. Symbolically we write

$$\int_{M\setminus B_R} |\Gamma||\pi| d\mu_g \le \left(\int_{M\setminus B_R} (|\Gamma|\rho^{1+\delta})^p \rho^{-3} d\mu_g\right)^{1/p} \cdot \left(\int_{M\setminus B_R} (|\pi|\rho^{2-\delta})^q \rho^{-3} d\mu_g\right)^{1/q}.$$

Now if $\delta > 1/2$ we have

$$\int_{M\setminus B_R} (|\pi|\rho^{2-\delta})^q \rho^{-3} d\mu_g \le \left(\int_{M\setminus B_R} (|\pi|\rho^{1+\delta})^p \rho^{-3} d\mu_g\right)^{q/p} \cdot \left(\int_{M\setminus B_R} \rho^{-\epsilon-3} d\mu_g\right)^{(p-q)/p}$$

where $\epsilon = (2\delta-1)pq/(p-q) > 0$. Thus we have

$$\left| P_i - \oint_{|\mathbf{x}|=R} \pi_{ij}\nu^j d\sigma_g \right| \le cR^{-\epsilon}$$

where the constant c depends only on the $W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$ norm of $(g - \delta, \pi)$. The continuity now follows from continuity of the approximating surface integral. This completes the proof of Theorem 1. q.e.d.

4. Solving the Deformation Problem

Let k be a nonnegative integer. Given a compactly contained domain $\Omega \subset M$, we let d denote the distance function to $\partial\Omega$. We let $\zeta \in C^\infty_c(\Omega)$ be a bump function (identically one on most of Ω), and let $\rho \le 1$ be a function which is positive in Ω and supported in $\overline{\Omega}$, and which is identical to d^N or $e^{-N/d}$ near $\partial\Omega$. We can require ρ to be C^k if Ω is a C^k-domain (for $N > k$ in case $\rho = d^N$), and using the exponentially decaying weight we can have ρ *smooth* for smooth Ω.

In this section we prove the following local deformation theorem for the Einstein constraints.

Theorem 2. *Let $\Omega \subset M$ be a compactly contained C^{k+2}-domain, and let $\zeta \in C^\infty_c(\Omega)$ be a bump function and $\rho \in C^{k+2}(M)$ be a weight function as above. Let g_0 be a $C^{k+4,\alpha}(M)$ metric and π_0 a $C^{k+3,\alpha}(M)$ symmetric $(2,0)$-tensor. Suppose that the linearization $D\Phi_{(g_0,\pi_0)}$ of the constraint map $\Phi : C^{k+2,\alpha}(\Omega) \times \mathcal{S}^{k+2,\alpha}_{(2,0)}(\Omega) \to C^{k,\alpha}(\Omega) \times \mathcal{X}^{k+1,\alpha}(\Omega)$ has an injective formal L^2-adjoint $D\Phi^*_{(g_0,\pi_0)}$ at (g_0,π_0), where we can consider*

$D\Phi^*_{(g_0,\pi_0)} : H^{2,1}_{\text{loc}}(\Omega) \to L^2_{\text{loc}}(\Omega)$. *Then for N sufficiently large, there is an $\epsilon > 0$ such that for any function $v \in C^{k,\alpha}(\overline{\Omega})$ and any vector field $W \in \mathcal{X}^{k+1,\alpha}(\overline{\Omega})$ for which $\big((v,W) - \Phi(g_0,\pi_0)\big) \in C^{k,\alpha}_{\rho^{-1}}(\Omega) \times \mathcal{X}^{k+1,\alpha}_{\rho^{-1}}(\Omega)$ with the support of $\big((v,W) - \Phi(g_0,\pi_0)\big)$ contained in $\overline{\Omega}$ and with $\|(v,W) - \Phi(g_0,\pi_0)\|_{C^{k,\alpha}_{\rho^{-1}}(\Omega)\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}(\Omega)} < \epsilon$, there is a $C^{k+2,\alpha}(M)$ metric g and a $C^{k+2,\alpha}(M)$ symmetric tensor π on M with $\Phi(g,\pi) = (v,W)$ in Ω and $(g,\pi) = (g_0,\pi_0)$ outside Ω. Moreover, $(g,\pi) \in C^{k+2,\alpha}(M) \times C^{k+2,\alpha}(M)$ depends continuously on $\big((v,W) - \Phi(g_0,\pi_0)\big) \in C^{k,\alpha}_{\rho^{-1}}(\Omega) \times \mathcal{X}^{k+1,\alpha}_{\rho^{-1}}(\Omega)$.*

If in addition $\big((v,W) - \Phi(g_0,\pi_0)\big) \in C^\infty_c(\Omega)$, and (g_0,π_0) and $\partial\Omega$ are smooth, and if we use an exponential weight, then we can solve for (g,π) smooth.

If the adjoint of the linearization has nontrivial kernel K, then K is finite-dimensional, and the analogous theorem holds for solving $\Phi(g,\pi) - (v,W) \in \zeta K$.

The analogous result for the scalar curvature operator is found in [**C**]; in particular it is shown there that at generic metrics g_0, for functions S so that the difference $S - R(g_0)$ vanishes outside Ω and is sufficiently small (in a weighted Hölder norm), there is a metric g with $R(g) = S$, so that $g - g_0$ is small and supported in $\overline{\Omega}$. The regularity statements are analogous to those above.

The proof of Theorem 2 will be carried out over the next few sections. For future use, if K is the kernel of $D\Phi^*_{(g_0,\pi_0)}$, we define S_g to be the $L^2(d\mu_g)$-orthogonal complement of ζK, and we take $S_0 = S_\delta$ where appropriate. We will emphasize the case we require to prove Theorem 4, for which we consider $(v,W) = 0$ and $(g_0,\pi_0) = (\delta, 0)$ is the Minkowski data. In this case K is ten-dimensional, and so we are solving the vacuum constraints up to the finite-dimensional error, i.e., $\Phi(g,\pi) \in \zeta K$.

4.1. The Basic Estimate. We prove an elliptic estimate in the ρ-weighted Sobolev spaces in a C^2-domain Ω. Since ρ decays at $\partial\Omega$, the boundary decay is imposed by ρ^{-1}, not ρ, in the sense that tensors in ρ^{-1}-weighted spaces will have to decay suitably at the boundary. We remark that we get a global estimate, even though the functions in the ρ-weighted spaces may not decay at the boundary. The key to achieving this estimate is using the overdetermined-ellipticity correctly. We first state the estimate in the form needed for the asymptotic gluing (Theorem 4), and later remark how to get the estimate required for Theorem 2.

Theorem 3. *Let Ω be a compactly contained C^2-domain in $\mathbb{R}^3$, and the weight ρ is defined as above. For N sufficiently large, there is a constant C and an $\epsilon > 0$ so that for all data (g,π) within ϵ of the*

Minkowski data $(\delta, 0)$ *in* $C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$*, and for all* $(f, X) \in S_g$*,*

$$\|(f, X)\|_{H^{2,1}_\rho(\Omega, d\mu_g)} \leq C \|D\Phi^*_{(g,\pi)}(f, X)\|_{L^2_\rho(\Omega, d\mu_g)}. \tag{12}$$

Proof. We first prove estimates on the vector field X. By Lemma 2.1 it suffices to prove these estimates for X a smooth vector field on $\overline{\Omega}$. Furthermore we define $\mathcal{D}_g X = L_X g$, to emphasize we are thinking of the operator acting on X.

Lemma 4.1. *For* ρ *as above with* N *sufficiently large, there is a constant* C *so that for all* X *orthogonal* (*or merely in some fixed subspace transverse*) *to the Killing fields of a metric* g*, the following estimate holds:*

$$\|X\|_{H^1_\rho(\Omega)} \leq C \|\mathcal{D}_g X\|_{L^2_\rho(\Omega)}. \tag{13}$$

Proof. We first note that since the Killing fields are Jacobi fields along geodesics, they are locally determined by their first order jet at a point, and so the space of Killing fields on Ω is finite-dimensional and the fields in the kernel will be smooth (as much as the smoothness of the metric and domain will allow) up to the boundary of Ω. Let $\Omega_\epsilon = \{x \in \Omega : d(x, \partial\Omega) > \epsilon\}$. The Lie derivative is an overdetermined-elliptic operator; that is, it has injective symbol. By standard elliptic theory, then, one gets an interior estimate of the form

$$\|X\|_{H^1(\Omega_0)} \leq C(\|\mathcal{D}_g X\|_{L^2(\Omega_\epsilon)} + \|X\|_{L^2(\Omega_\epsilon)}) \tag{14}$$

where $\Omega_0 \subset\subset \Omega$ and C depends on $d(\Omega_0, \partial\Omega)$, but is independent of $\epsilon < \frac{1}{2} d(\Omega_0, \partial\Omega)$ small. To get a global estimate we foliate Ω near $\partial\Omega$ by level sets $\Sigma_r = \{x \in \Omega : d(x, \partial\Omega) = r\}$ of the distance function to $\partial\Omega$; there is an $r_0 > 0$ depending on Ω for which these level sets are regular hypersurfaces. By standard elliptic theory on these closed hypersurfaces Σ we get

$$\|X_\Sigma\|_{H^1(\Sigma)} \leq C \left(\|\mathcal{D}_g X\|_{L^2(\Sigma)} + \|X\|_{L^2(\Sigma)}\right)$$

since the difference between $(\mathcal{D}_g X)|_{T\Sigma}$ and $\mathcal{D}_{g_\Sigma} X_\Sigma$ is zeroth-order in X; g_Σ is the induced metric, and X_Σ is the projection of X onto $T\Sigma$. The constant C in this estimate can be taken uniform in the distance to $\partial\Omega$ sufficiently small. Recall that since $|\nabla d(\cdot, \partial\Omega)| = 1$, the co-area formula gives for positive functions or L^1-functions G: $\int\limits_{r_1 < d < r_2} G \, d\mu_g = \int\limits_{r_1}^{r_2} \int\limits_{\Sigma_r} G \, d\sigma_g \, dr$. Applying that here yields

$$\|(X^T, \nabla_\Sigma X^T)\|_{L^2(A)} \leq C \left(\|\mathcal{D}_g X\|_{L^2(A)} + \|X\|_{L^2(A)}\right)$$

where $X^T = X - \langle X, \nabla d\rangle \nabla d$ is the vector field on $A = \{x \in \Omega : r_1 < d(x, \partial\Omega) < r_2\}$ given by the projections X_Σ. Since the constant is uniform in r_1, we see that combining with the interior estimate we get

$$\|(X^T, \nabla_\Sigma X^T)\|_{L^2(\Omega_\epsilon)} \le C\left(\|\mathcal{D}_g X\|_{L^2(\Omega_\epsilon)} + \|X\|_{L^2(\Omega_\epsilon)}\right) \tag{15}$$

where C is uniform in ϵ small. We can use the co-area formula to integrate the square of this estimate against $\rho'(\epsilon)$ (which is positive for ϵ small), and integrate by parts to get the desired weighted estimate, as in [**C**].

It remains to estimate the terms involving components of X and ∇X normal to Σ. We compute in a local orthonormal frame e_1, e_2, e_3 adapted to Σ; we take $e_3 = \nabla d$, well-defined in a neighborhood of $\partial\Omega$. We have estimates for $X_{i;j}$ for $i, j = 1, 2$, and hence by taking the trace of $\mathcal{D}_g X$, we can also estimate $X_{3;3}$ as above. It suffices then to estimate $X_{i;3}$ for $i = 1, 2$. Let E_i denote a set of the form $\{x \in \Omega : 0 < d(x, \partial\Omega) < r_i\}$, and let ξ be a cut-off function of the distance to the boundary, which is identically one for $d < r_2 < r_1 \le 1$, and vanishes outside E_1. We first note

$$\begin{aligned}2\int_{E_2}\sum_{i=1}^{2}(X_{i;3}^2 + X_{3;i}^2)\rho\, d\mu_g &\le 2\int_{\Omega}\xi\sum_{i=1}^{2}(X_{i;3}^2 + X_{3;i}^2)\rho\, d\mu_g \\ &\le \int_{\Omega}|\mathcal{D}_g X|^2\rho\, d\mu_g - 4\int_{E_1}\xi\sum_{i=1}^{2}X_{i;3}X_{3;i}\rho\, d\mu_g.\end{aligned}$$

We now want to integrate by parts to make the latter integrand $X_{i;3i}X_3$. We note that by the arithmetic-geometric mean inequality (AM-GM) $ab \le \frac{1}{2}\left(\epsilon^2 a^2 + \frac{b^2}{\epsilon^2}\right)$, terms in the integrand of the form $X_{i;3}X_j$ or $X_{3:i}X_j$ can be replaced (up to a constant factor) by $|X|^2$, with the other term being absorbed on the left side of the above inequality. This observation allows us to switch between covariant derivatives (on Σ and Ω) and partial derivatives as needed. By integration by parts (divergence theorem on Σ) and the co-area formula, and using AM-GM as needed, we get (since for $i = 1, 2$, $\zeta_{,i} = 0 = \rho_{,i}$)

$$\begin{aligned}\frac{3}{2}\int_{E_2}\sum_{i=1}^{2}(X_{i;3}^2 + X_{3;i}^2)\rho\, d\mu_g &\le \int_{\Omega}|\mathcal{D}_g X|^2\rho\, d\mu_g \\ &\quad + 4\int_{\Omega}\xi\sum_{i=1}^{2}X_{i;3i}X_3\rho\, d\mu_g + C\|X\|^2_{L^2_\rho(\Omega)}.\end{aligned}$$

We now use the Ricci formula to commute covariant derivatives: $X_{i;jk} - X_{i;kj} = X_l R^l_{kji}$. So we now have

$$\frac{3}{2}\int\limits_{E_2}\sum_{i=1}^{2}(X_{i;3}^2 + X_{3;i}^2)\rho d\mu_g \leq \int\limits_{\Omega}|\mathcal{D}_g X|^2\rho d\mu_g + 4\int\limits_{\Omega}\xi\sum_{i=1}^{2}X_{i;i3}X_3\rho d\mu_g + C\|X\|^2_{L^2_\rho(\Omega)}.$$

If we integrate by parts again (this time on Ω), we get (since the boundary integrals vanish)

$$\begin{aligned}\int\limits_{E_2}\sum_{i=1}^{2}(X_{i;3}^2 + X_{3;i}^2)\rho d\mu_g &\leq \int\limits_{\Omega}|\mathcal{D}_g X|^2\rho d\mu_g - 4\int\limits_{\Omega}\xi\sum_{i=1}^{2}X_{i;i}X_{3;3}\rho d\mu_g \\ &\quad - 4\int\limits_{\Omega}(\xi\rho)_{,3}\sum_{i=1}^{2}X_{i;i}X_3 d\mu_g + C\|\nabla_\Sigma X^T\|^2_{L^2_\rho(\Omega)} + C\|X\|^2_{L^2_\rho(\Omega)}.\end{aligned}$$

We can estimate the second and third summands on the right side above by the AM-GM inequality together with the weighted estimate on $(X^T, \nabla_\Sigma X^T)$, bounding them by (in case $\rho = d^N$ near the boundary; a similar argument holds for ρ exponentially decaying)

$$C\int\limits_{\Omega}|\mathcal{D}_g X|^2\rho d\mu_g + C\|X\|^2_{L^2_\rho(\Omega)} + \int\limits_{\Omega}|X|^2 d^{-2}\rho d\mu_g.$$

We now claim there is a constant C (depending on N, as well as g and Ω) so that

$$\int\limits_{\Omega}|X|^2 d^{-2}\rho d\mu_g \leq C\int\limits_{\Omega}|\mathcal{D}_g X|^2\rho d\mu_g.$$

If this were not the case, we could find a sequence X_k of vector fields so that

$$\int\limits_{\Omega}|X_k|^2 d^{-2}\rho d\mu_g = 1$$

but for which

$$\int\limits_{\Omega}|\mathcal{D}_g X_k|^2\rho d\mu_g \to 0.$$

Now by the previous estimates, then,

$$\int_\Omega |\nabla_g X_k|^2 \rho d\mu_g \leq C\left(\int_\Omega |\mathcal{D}_g X_k|^2 \rho d\mu_g + \int_\Omega |X_k|^2 \rho d\mu_g\right) + 1. \tag{16}$$

Thus we see that $X_k \rho^{1/2}$ is bounded in $H^1(\Omega)$, and so we can assume (by taking a subsequence) that there is a vector field X so that $X_k \rho^{1/2}$ converges weakly in H^1 and strongly in L^2 to $X\rho^{1/2}$. So X is in $H^1_{loc}(\Omega)$, and X_k converges to X locally in L^2, so that $\mathcal{D}_g X = 0$ weakly. But the X_k are transverse to the Killing fields, and so $X = 0$, and thus $X_k \rho^{1/2} \to 0$ in L^2. Now we note that there is an r_1 and a constant C (independent of N) so that on E_1,

$$\Delta_g \rho \geq CN^2 d^{-2} \rho$$

(again, in case ρ is exponentially decaying we have a similar estimate). So we get by integration by parts

$$CN^2 \int_\Omega \xi |X_k|^2 d^{-2} \rho d\mu_g \leq 2\int_\Omega \xi |X_k||\nabla_g X_k| N d^{-1} \rho d\mu_g + \int_\Omega \xi' |X_k|^2 N d^{-1} \rho d\mu_g. \tag{17}$$

Since ξ' is compactly supported in Ω, the second integral on the right side goes to zero in k. By Cauchy-Schwarz we also have

$$2\int_\Omega \xi |X_k||\nabla_g X_k| N d^{-1} \rho d\mu_g \leq 2N\left(\int_\Omega \xi |\nabla_g X_k|^2 \rho d\mu_g\right)^{\frac{1}{2}}\left(\int_\Omega \xi |X_k|^2 d^{-2} \rho d\mu_g\right)^{\frac{1}{2}}.$$

Now using the previous estimates (16), (17), we get that

$$CN^2 \int_\Omega \xi |X_k|^2 d^{-2} \rho d\mu_g \leq 2N(1 + o(1)) + No(1)$$

where $o(1)$ denotes a function which goes to zero as $k \to \infty$. So for N sufficiently large, we have for large k

$$\int_\Omega \xi |X_k|^2 d^{-2} \rho d\mu_g \leq \frac{1}{2}.$$

But we also have by the L^2_{loc}-convergence that

$$\lim_{k\to+\infty} \int_\Omega (1-\xi)|X_k|^2 d^{-2} \rho d\mu_g = 0.$$

So far large enough k,

$$\int_\Omega |X_k|^2 d^{-2} \rho d\mu_g < 1$$

which is a contradiction. q.e.d.

We continue the proof of the Basic Estimate, Theorem 3. As we noted above (Lemma 2.5), the kernel of $D\Phi^*_{(\delta,0)}$ is $K_0 \oplus K_1$. $(f, X) \in S_0$ precisely if f is orthogonal to ζK_0 and X is orthogonal to ζK_1. By the preceding lemma, then, to prove the Basic Estimate at the Minkowski data, we now just need to estimate f by $L^*_\delta f$. We do have an estimate of the form

$$\|f\|_{H^2(\Omega)} \leq C \left(\|L^*_g f\|_{L^2(\Omega)} + \|f\|_{L^2(\Omega)} \right).$$

Indeed using Eq. (2.2) and taking a trace, we see that

$$\mathrm{Hess}_g f = -\frac{1}{2} \left(Tr_g(L^*_g f) + f R(g) \right) g - f \,\mathrm{Ric}\,(g).$$

Since f is orthogonal to ζK_0, we can apply standard theory (basically the Rellich theorem) to get the estimate

$$\|f\|_{H^2(\Omega_\epsilon)} \leq C \|L^*_g f\|_{L^2(\Omega_\epsilon)} \tag{18}$$

where C is uniform in ϵ small, and so we get the desired weighted estimate as indicated above following (15).

We now have the following estimate for $(f, X) \in S_0$:

$$\|(f, X)\|_{H^{2,1}_\rho(\Omega,\delta)} \leq C \|D\Phi^*_{(\delta,0)}(f, X)\|_{L^2_\rho(\Omega,\delta)}. \tag{19}$$

For data (g, π) close to the Minkowski data in $C^2(\overline{\Omega}) \times C^1(\overline{\Omega})$, Theorem 3 follows by perturbation. q.e.d.

4.2. Variational Method. In this section we show how solutions to $\Pi_{S_0} D\Phi_{(g,\pi)}(h, \omega) = (\phi, V)$ for $(\phi, V) \in L^2_{\rho^{-1}}(\Omega) \cap S_0$, (and (g, π) close to Minkowski data), can be obtained from standard variational arguments.

We let Π_{S_0} denote the orthogonal projection onto S_0 with respect to the metric δ, and we define the operator $\mathcal{R}_{(g,\pi)}$ to be the linearization of the map $\Pi_{S_0} \circ \Phi$, so that $\mathcal{R}_{(g,\pi)} = \Pi_{S_0} \circ D\Phi_{(g,\pi)}$. We want to show this map surjects from a suitable space onto $S_0 \cap L^2_{\rho^{-1}}(\Omega)$. In fact it is easy to see this will be true if we can show the map $\mathcal{P}_{(g,\pi)} := \Pi_{S_g} \circ D\Phi_{(g,\pi)}$ is surjective to $S_g \cap L^2_{\rho^{-1}}(\Omega)$, for (g, π) near the Minkowski data.

We define the formal adjoint $\mathcal{P}^*_{(g,\pi)}$ with respect to the metric g, and for $\rho(f, X) \in L^2_{\rho^{-1}}(\Omega) \cap S_g$ we define the functional $\mathcal{G}$ by

$$\mathcal{G}(u, Z) = \int_\Omega \left(\frac{1}{2} \mid \mathcal{P}^*_{(g,\pi)}(u, Z) \mid^2 - (f, X) \cdot_g (u, Z) \right) \rho \, d\mu_g. \tag{20}$$

We consider the infimum $\mu = \mu_{(f,X)} \leq 0$ over all $(u, Z) \in \mathcal{V}_g = H^{2,1}_\rho(\Omega) \cap S_g$. For any $(u, Z) \in \mathcal{V}_g$, and any (ψ, W) with compact support in Ω, we have

$$\begin{aligned}(21) \qquad \langle \mathcal{P}^*_{(g,\pi)}(u, Z), (\psi, W)\rangle_{L^2(d\mu_g)} &= \langle (u, Z), \mathcal{P}_{(g,\pi)}(\psi, W)\rangle_{L^2(d\mu_g)} \\ &= \langle (u, Z), D\Phi_{(g,\pi)}(\psi, W)\rangle_{L^2(d\mu_g)} \\ &= \langle D\Phi^*_{(g,\pi)}(u, Z), (\psi, W)\rangle_{L^2(d\mu_g)}\end{aligned}$$

where the second inequality follows from $(u, Z) \in \mathcal{V}_g$. So on $\mathcal{V}_g$ we have $D\Phi^*_{(g,\pi)} = \mathcal{P}^*_{(g,\pi)}$, and thus the Basic Estimate yields a minimizer to our variational problem, as we now recall. We first note that μ is finite.

Lemma 4.2. *For any $(f, X) \in L^2_\rho(\Omega)$, $\mu > -\infty$.*

Proof. We simply note that the Basic Estimate, along with Cauchy-Schwarz, yields the estimate

$$\mathcal{G}(u, Z) \geq \frac{1}{2C}\|(u, Z)\|^2_{H^{2,1}_\rho(\Omega)} - \|(f, X)\|_{L^2_\rho(\Omega)}\|(u, Z)\|_{H^{2,1}_\rho(\Omega)}.$$

q.e.d.

Corollary 4.3. *For any $(f, X) \in L^2_\rho(\Omega)$, $\mu = \lim_{i\to\infty} \mathcal{G}(u_i, Z_i)$ for some sequence $\{u_i, Z_i\}$ with $\{\|(u_i, Z_i)\|_{H^{2,1}_\rho(\Omega)}\}$ bounded.*

Standard functional analysis now gives the existence of a minimizer (u_0, Z_0) in the Hilbert space $\mathcal{V}_g$ [**C**]. In fact by the convexity of the functional $\mathcal{G}$ the minimizer is unique. In any case, since the orthogonal complement of S_g is composed of $C^\infty_c(\Omega)$-functions, we see that the Euler-Lagrange equations will hold on all of $C^\infty_c(\Omega)$, yielding the weak formulation of

$$\mathcal{P}_{(g,\pi)}\rho\mathcal{P}^*_{(g,\pi)}(u_0, Z_0) = \Pi_{S_g} D\Phi_{(g,\pi)}\rho D\Phi^*_{(g,\pi)}(u_0, Z_0) = \rho(f, X).$$

A simple argument comparing the L^2-projections shows that $\rho D\Phi^*_{(g,\pi)}(u_0, Z_0)$ is also a weak solution of the equation

$$\Pi_{S_0} D\Phi_{(g,\pi)}\rho D\Phi^*_{(g,\pi)}(u_0, Z_0) = (\phi, V)$$

where $(\phi, V) \in S_0 \cap L^2_{\rho^{-1}}(\Omega)$ and where we take $\mathcal{P}_{(g,\pi)}\rho D\Phi^*_{(g,\pi)}(u_0, Z_0) = \rho(f, X)$ to be the $L^2(d\mu_g)$-projection of (ϕ, V) to S_g.

4.3. Pointwise estimates and the nonlinear problem. Assuming we start with tensors (g_0, π_0) on $\mathbb{R}^3$, sufficiently smooth and sufficiently close to Minkowski data on $\Omega \subset\subset \mathbb{R}^3$, and which solve the constraint equations outside Ω (and in a neighborhood of the boundary), we produce tensors $(\bar{g}, \bar{\pi})$ which agree with the original data outside Ω and which satisfy $\Pi_{S_0}\Phi(\bar{g}, \bar{\pi}) = 0$. The basic procedure is straightforward: we use the variational procedure to solve the problem at the linear level, and we iterate the process of correction. In this section we write down the estimates needed to show convergence of the iteration.

We start by solving the equation $\Pi_{S_0} D\Phi_{(g_0,\pi_0)}\rho D\Phi^*_{(g_0,\pi_0)}(u_0, Z_0) = -\Pi_{S_0}\Phi(g_0,\pi_0)$ as above. Assuming $(g_0,\pi_0) \in \mathcal{M}^{k+4,\alpha}(\overline{\Omega}) \times \mathcal{S}^{k+3,\alpha}_{(2,0)}(\overline{\Omega})$, we get regularity and interior elliptic estimates on the solution (u_0, Z_0). We will invoke the weighting scheme for elliptic systems of mixed orders due to Douglis and Nirenberg [**DN**]. Note that for notational simplicity we will omit the subscript (g_0,π_0) on the operators below.

Now the operator $\rho^{-1}D\Phi\rho D\Phi^*$ is uniformly elliptic, and gives a 4×4 system, which we will symbolically write as $L_j U = L_{jk}(D)U^k$, and the weights s_j and t_k are defined so that the order of L_{jk} is s_j+t_k; the elliptic estimate comes from bounding the Hölder semi-norm $\sum_{i=1}^{4}\left[U^i\right]_{t_i,\alpha}$ up to a constant factor by $\sum_{i=1}^{4}[L_i U]_{-s_i,\alpha}$. In our case U stands for (u, Z), and so we want the weights to satisfy the following conditions: $s_1 + t_1 = 4$; for $j, k > 1$, $s_1 + t_k = 3 = s_j + t_1$ and $s_j + t_k = 2$. With this in mind we let $t_1 = 4$, and the other $t_k = 3$, we let $s_1 = 0$ and the other $s_j = -1$. Then for $\Omega' \subset\subset \Omega$

$$\begin{aligned}
&\|(h_0,\omega_0)\|_{k+2,\alpha,\Omega'} \\
&= \|\rho D\Phi^*(u_0,Z_0)\|_{k+2,\alpha,\Omega'} \leq C\left(\|u_0\|_{k+4,\alpha,\Omega'} + \|Z_0\|_{k+3,\alpha,\Omega'}\right) \\
&\leq C\Big(\|(u_0,Z_0)\|_{L^2_\rho(\Omega)} + \|D\mathcal{H}(\rho D\Phi^*(u_0,Z_0))\|_{k,\alpha,\Omega} \\
&\quad + \|D\mathrm{div}\,(\rho D\Phi^*(u_0,Z_0))\|_{k+1,\alpha,\Omega}\Big).
\end{aligned}$$

If the weight ρ is d^N near the boundary, then the interior estimates local to the boundary ($B' = B(x, \frac{d(x)}{4}) \subset B = B(x, \frac{d(x)}{2})$) are

$$\begin{aligned}
\|(h_0,\omega_0)\|_{k+2,\alpha,B'} \leq C d^{N-\psi(k,\alpha)}\Big(&\|(u_0,Z_0)\|_{L^2(B)} \\
&+ \|\rho^{-1}D\mathcal{H}(\rho D\Phi^*(u_0,Z_0))\|_{k,\alpha,B} \\
&+ \|\rho^{-1}D\mathrm{div}\,(\rho D\Phi^*(u_0,Z_0))\|_{k+1,\alpha,B}\Big).
\end{aligned}$$

Here $\psi(k,\alpha)$ is linear in k and α, and also note that the lower-order coefficients in the operator $\rho^{-1}D\Phi\rho D\Phi^*$ have powers of $1/d$ that are accommodated by scaling in the estimate, or can also be subsumed into $\psi(k,\alpha)$. Using the estimate

$$\|(u_0,Z_0)\|_{L^2(B)} \leq C d^{-\frac{N}{2}}\|u_0,Z_0)\|_{L^2_\rho(\Omega)}$$

we then have

$$\begin{aligned}(22)\quad &\|(h_0,\omega_0)\|_{k+2,\alpha,B'}\\ &\le Cd^{\frac{N}{2}-\psi(k,\alpha)}\Big(\|(u_0,Z_0)\|_{L^2_\rho}+\|\rho^{-\frac{1}{2}}D\mathcal{H}(\rho D\Phi^*(u_0,Z_0))\|_{k,\alpha,B}\\ &\quad+\|\rho^{-\frac{1}{2}}D\mathrm{div}\,(\rho D\Phi^*(u_0,Z_0))\|_{k+1,\alpha,B}\Big).\end{aligned}$$

In fact we need to do the above estimates in terms of the projected operator. Because the difference between the operators $\rho^{-1}D\Phi\rho D\Phi^*$ and $\rho^{-1}\mathcal{R}\rho D\Phi^*$ on $\mathcal{V}_g$ (see (21)) is finite-dimensional, we get the following interior estimate

$$\begin{aligned}(23)\quad &\|(h_0,\omega_0)\|_{k+2,\alpha,\Omega'}\\ &=\|\rho D\Phi^*(u_0,Z_0)\|_{k+2,\alpha,\Omega'}\le C\left(\|u_0\|_{k+4,\alpha,\Omega'}+\|Z_0\|_{k+3,\alpha,\Omega'}\right)\\ &\le C\left(\|(u_0,Z_0)\|_{L^2_\rho(\Omega)}+\|\mathcal{R}\rho D\Phi^*(u_0,Z_0)\|_{C^{k,\alpha}_{\rho^{-1}}(\Omega)\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}(\Omega)}\right).\end{aligned}$$

Here Ω' is taken to be a "large" fixed domain containing the support of the fields in ζK. We have used that there is a constant C so that for all $(\phi,V)\in\zeta K$

$$\|(\phi,V)\|_{C^{k,\alpha}(\Omega)\times\mathcal{X}^{k+1,\alpha}(\Omega)}\le C\|(\phi,V)\|_{L^2(\Omega')}$$

in particular for (ϕ,V) of the form $\Pi_{\zeta K}D\Phi\rho D\Phi^*(u,Z)$. We then use that by compactness, for all $\epsilon>0$ there is a $C(\epsilon)$ so that

$$\begin{aligned}&\|(u,Z)\|_{C^4(\Omega')\times\mathcal{X}^3(\Omega')}\\ &\le\epsilon\|(u,Z)\|_{C^{k+4,\alpha}(\Omega')\times\mathcal{X}^{k+3,\alpha}(\Omega')}+C(\epsilon)\|(u,Z)\|_{L^2(\Omega')}.\end{aligned}$$

Since the fields in ζK are supported away from the boundary, $\mathcal{R}=D\Phi$ near the boundary. Thus the estimates we have local to the boundary are as in (23), except for the appearance of the factor $d^{\frac{N}{2}-\psi(k,\alpha)}$ on the right-hand side, which yields the decay of the deformation tensors at $\partial\Omega$.

Next we note that we can bound (h_0,ω_0) in $L^2_{\rho^{-1}}(\Omega)$ by the basic injectivity estimate for $D\Phi^*$, and we can bound the lower order term in the estimate by the variational inequality $\mathcal{G}(u_0,Z_0)\le 0$, obtaining

$$\begin{aligned}\|(h_0,\omega_0)\|_{L^2_{\rho^{-1}}}\le C\|(u_0,Z_0)\|_{L^2_\rho}&\le C\|\Pi_{S_0}\Phi(g_0,\pi_0)\|_{L^2_{\rho^{-1}}}\\ &\le C\|\Phi(g_0,\pi_0)\|_{L^2_{\rho^{-1}}}.\end{aligned}$$

This inequality can be inserted into the decay estimates near the boundary (22), and it can also be used now to get a global estimate on (h_0,ω_0):

$$(24)\qquad \|(h_0,\omega_0)\|_{L^2_{\rho^{-1}}}+\|(h_0,\omega_0)\|_{k+2,\alpha}\le C\|\Phi(g_0,\pi_0)\|_{C^{k,\alpha}_{\rho^{-1}}\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}}.$$

We now iterate the process of linear correction. We linearize only about the initial metric and momentum tensor, since the coefficients in the fourth-order system depend on derivatives of (g_0, π_0); for example in computing $D\Phi_{(g_0,\pi_0)}\rho D\Phi^*_{(g_0,\pi_0)}$, one differentiates the Ricci curvature twice, which involves four derivatives of the metric (*cf.* Lemmas 2.2 and 2.3). The above estimates show a gain of two derivatives in the metric deformation tensor, not four. Hence we cannot apply Newton's method to produce a solution. However, we can still produce solutions by linear correction, as we now describe. We use Taylor's formula in the form

$$\Phi(g_0 + h, \pi_0 + \omega) = \Phi(g_0, \pi_0) + D\Phi_{(g_0,\pi_0)}(h, \omega) + O(\|(h, \omega)\|^2_{k+2,\alpha})$$

where

$$\|O(\|(h, \omega)\|^2_{k+2,\alpha})\|_{C^{k,\alpha}\times\mathcal{X}^{k+1,\alpha}} \leq C\|(h, \omega)\|^2_{k+2,\alpha}$$

and the constant C can be taken uniformly for data near (g_0, π_0) and on subdomains of Ω. We also have the following Taylor's theorem for the projected operator on Ω:

$$\begin{aligned} \Pi_{S_0}\Phi(g_0 + h, \pi_0 + \omega) \\ = \Pi_{S_0}\Phi(g_0, \pi_0) + \mathcal{R}_{(g_0,\pi_0)}(h, \omega) + O(\|(h, \omega)\|^2_{k+2,\alpha}). \end{aligned} \tag{25}$$

By solving the equation

$$\Pi_{S_0} D\Phi_{(g_0,\pi_0)}\rho D\Phi^*_{(g_0,\pi_0)}(u_0, Z_0) = -\Pi_{S_0}\Phi(g_0, \pi_0)$$

as above, then, we have that

$$\|\Pi_{S_0}\Phi(g_1, \pi_1)\|_{C^{k,\alpha}\times\mathcal{X}^{k+1,\alpha}} \leq C(\|(h_0, \omega_0)\|^2_{k+2,\alpha})$$

where $(h_0, \omega_0) = \rho D\Phi^*_{(g_0,\pi_0)}(u_0, Z_0)$ and $(g_1, \pi_1) = (g_0, \pi_0) + (h_0, \omega_0)$; this estimate holds locally near the boundary as well, outside the support of ζ.

At the next stage, we again use the linearization at (g_0, π_0) (as explained above) and solve $\Pi_{S_0} D\Phi\rho D\Phi^*(u_1, Z_1) = -\Pi_{S_0}\Phi(g_1, \pi_1)$. We obtain $(h_1, \omega_1) = \rho D\Phi^*(u_1, Z_1)$ and $(g_2, \pi_2) = (g_1, \pi_1) + (h_1, \omega_1)$. The quadratic decay at the first step does not propagate (since we linearize about the initial data only) but the estimates above show that a Picard iteration converges; indeed it is straightforward to check that the error terms we get by using the fixed linear operator to correct the nonlinear term at each stage still allow the iteration to converge geometrically at a rate better than linear, but worse than quadratic. In fact we have the following iteration lemma, a straightforward adaptation of Prop. 3.9 from [**C**]. (We remark that there are a few harmless typos in [**C**]; in particular, $C^{k,\alpha}_{\rho^{-1}}$ should be defined as we have done here, and so the norm one uses on (h, ω) (simply h in [**C**]) in the Picard iteration is the norm on the left side of (26) below.)

Lemma 4.4. *Let C be as in* (24). *Suppose we have recursively obtained $(h_0, \omega_0), \ldots, (h_{m-1}, \omega_{m-1})$ so that $g_0, \ldots, g_m$ are $C^{k+2,\alpha}$-metrics.*

Suppose there is a constant K and a $\delta \in (0,1)$ so that for all $l < m$,

$$\|(h_l,\omega_l)\|_{L^2_{\rho^{-1}}} + \|(h_l,\omega_l)\|_{k+2,\alpha} \le CK\|\Phi(g_0,\pi_0)\|^{1+l\delta}_{C^{k,\alpha}_{\rho^{-1}}\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}}$$

and for all $j \le m$

$$\|\Pi_{S_0}\Phi(g_j,\pi_j)\|_{C^{k,\alpha}_{\rho^{-1}}\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}} \le K\|\Phi(g_0,\pi_0)\|^{1+j\delta}_{C^{k,\alpha}_{\rho^{-1}}\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}}.$$

Then for sufficiently small $\|\Phi(g_0,\pi_0)\|_{C^{k,\alpha}_{\rho^{-1}}\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}}$ (independent of m) the iteration can proceed, and the above inequalities persist.

The local estimates ((22), and the local version of (23)) near the boundary (outside the support of ζ) allow us to show the limiting tensors can be made to decay as much as we like as we approach $\partial\Omega$ by choosing N large enough; we can even get all derivatives decaying if we choose smooth data $\Phi(g_0,\pi_0)$ sufficiently small, supported away from $\partial\Omega$, and if we use an exponential weight ρ. Moreover we have a bound on the limiting tensors $(h,\omega) = \sum_{k=0}^{\infty}(h_k,\omega_k)$:

$$\|(h,\omega)\|_{L^2_{\rho^{-1}}} + \|(h,\omega)\|_{k+2,\alpha} \le C\|\Phi(g_0,\pi_0)\|_{C^{k,\alpha}_{\rho^{-1}}\times\mathcal{X}^{k+1,\alpha}_{\rho^{-1}}}. \tag{26}$$

This bound comes straight from the estimates done above (and the geometric convergence).

We remark that for the smooth case we solve the nonlinear equation in some finite regularity class to start, but using an exponentially decaying weight. We use standard bootstrapping on the quasilinear elliptic condition $\Phi((g_0,\pi_0)+\rho D\Phi^*(u,Z)) \in \zeta K \subset C^\infty$ to see that the solution $\rho D\Phi^*(u,Z)$ is smooth on the interior, and then use the decay of ρ to prove that all the derivatives decay near the boundary; we note that we have assumed that $\Phi(g_0,\pi_0)$ is supported away from the boundary, and we are considering the case $(v,W)=0$ of Theorem 2.

4.4. The General Case of Theorem 2. We now indicate how to finish the proof of the full version of Theorem 2. We need only prove the Basic Estimate (12) in the general case (i.e., not just for perturbations of the Minkowksi data), as the analogous variational method and nonlinear iteration will proceed as above.

The general version of the Basic Estimate is proved by first observing that the inequality $\|X\|_{H^1_\rho(\Omega)} \le C\big(\|\mathcal{D}_g X\|_{L^2_\rho(\Omega)} + \|X\|_{L^2_\rho(\Omega)}\big)$ follows by combining Lemma 4.1 with the fact mentioned earlier that the kernel of $\mathcal{D}_g$ is finite-dimensional and smooth up to the boundary. The condition that (f,X) is in the kernel of $D\Phi^*$ can be written (using equation (9)) as a second-order system of ordinary differential equations along geodesics, and so the kernel is also finite-dimensional and smooth (as smooth as allowed by the other data). Elementary manipulation using Lemma 2.3 then yields the estimate

$$\|(f,X)\|_{H^{2,1}_\rho(\Omega)} \le C\Big(\|D\Phi^*_{(g,\pi)}(f,X)\|_{L^2_\rho(\Omega)} + \|(f,X)\|_{L^2_\rho(\Omega)}\Big).$$

We can remove the lower-order term in the estimate by essentially the usual Rellich argument; to do this, one first shows the left side of the above estimate can be promoted to $\|(f,X)\rho^{\frac{1}{2}}\|_{H^{2,1}(\Omega)}$. This estimate incorporates different weights for different derivatives. In fact we already have the required estimate for the vector field X above, and the analogous estimate for f in terms of $L_g^* f$ is a straightforward modification of that argument. This completes the proof of Theorem 2. q.e.d.

5. Handling the cokernel

We apply the preceding analysis to solve the constraint equations with given model near infinity up to a finite-dimensional obstruction. Given any AF solution of the constraints on $\mathbb{R}^3$, we fix AF coordinates at infinity, and at a sufficiently large radius R we smoothly patch (using a smooth cut-off function whose k^{th} derivative is $O(R^{-k})$) our original data to the model data, the transition occurring in the annulus A_R from R to $2R$. To apply the preceding section, we will pullback and scale our glued data to the unit annulus A_1. Under this scaling the relevant geometric quantities (derivatives, curvatures) are $O(R^{-1})$, the (AC) conditions hold to order $O(R^{-2})$, and the vacuum constraint equations are scale-invariant. For large enough R the data will be sufficiently close to the flat data on the fixed annulus A_1, so the data can be perturbed to make the constraint functions lie in ζK on A_1.

It is at this point we need to be clear about what are suitable families of solutions to glue on near infinity. We define a family of solutions on the exterior of a fixed ball and smoothly parametrized on an open set $\mathcal{O} \subset \mathbb{R}^{10}$ to be *admissible* if, with reference to a fixed coordinate chart near infinity, the family satisfies (AC) locally uniformly with respect to the parameters, and the map $\Theta : \mathcal{O} \to \mathbb{R}^{10}$ which associates to each member of the family its energy-momenta $(E, \mathbf{P}, \mathbf{J}, \mathbf{C})$ is a homeomorphism onto an open subset of $\mathbb{R}^{10}$.

We note that slices in Kerr form an admissible family, parametrized by the total mass m, the angular momentum parameter a, and an element in the Poincaré group to control Euclidean motions of the AF coordinate system as well as boosts. It is well-known that under the action of the Poincaré group the energy-momentum transforms as a four-vector, and the center and angular momentum enjoy a similar transformation rule (they comprise a skew-symmetric $M_{\mu\nu}$ with M_{0k} a constant times C^k) cf. [**ADM**], [**BO**], [**RT**], which allows one to see that varying the slices near the given one gives a local homeomorphism from a ten-dimensional family of slices to the ten-dimensional space of energy-momenta. Note that time translations and rotations about the axis of symmetry of the Kerr are isometries, so we really mod out by the closed two-parameter subgroup they generate to parametrize the slices, leaving eight effective parameters from the group. Please see [**CD**] for an explicit formula and other examples.

Let $E_R \subset M$ correspond the the exterior $\{x \in \mathbb{R}^3 : |x| > R\}$ in an AF chart. We now state the gluing theorem.

Theorem 4. *Let (g, π) solve the vacuum constraint equations on M, and satisfy (AC) in an AF coordinate chart in a given end. Let $\mathcal{O}$ parametrize an admissible family of solutions, with $\lambda_0 \in \mathcal{O}$ so that $\Theta(\lambda_0)$ is the energy-momenta of (g, π). There is a radius R and a solution $(\overline{g}, \overline{\pi})$ of the constraints so that $(\overline{g}, \overline{\pi}) = (g, \pi)$ on $M \setminus E_R$, and $(\overline{g}, \overline{\pi})$ agrees with a suitably chosen member of the admissible family on E_{2R}.*

Coupled with Theorem 1, we have the following approximation result.

Theorem 5. *Let (g, π) be any AF solution of the vacuum constraints. Given any $\epsilon > 0$, there is a solution $(\overline{g}, \overline{\pi})$ within ϵ of (g, π) in the $W^{2,p}_{-\delta} \times W^{1,p}_{-1-\delta}$ norm, whose ADM energy-momentum $(E, \mathbf{P})$ is within ϵ of that of (g, π), and so that near infinity, $(\overline{g}, \overline{\pi})$ agrees with a member of an admissible asymptotic model family.*

Proof of Theorem 4. For sufficiently large R, we have a continuous map $\mathcal{I}$ from an open set $\mathcal{O} \subset \mathbb{R}^{10}$ to $\mathbb{R}^{10}$ as follows: take data corresponding to $\lambda \in \mathcal{O}$, glue it to the given data in A_R, and then perturb using the above techniques to data $(\bar{g}, \bar{\pi})$ with $\Phi(\bar{g}, \bar{\pi})$ lying in a fixed ten-dimensional vector space; we let $\mathcal{I}(\lambda)$ correspond to $\Phi(\bar{g}, \bar{\pi})$. By design, $(\bar{g}, \bar{\pi})$ is identical to the member of the asymptotic model family in the exterior end E_{2R}. In the next section we show that this map $\mathcal{I}$ has a zero near the parameter $\lambda_0 \in \mathcal{O}$ corresponding to the given initial AF data (g, π) with which we started, and thus this data solves the vacuum constraints. Notice that we cannot arbitrarily glue on anything we like, but the procedure finds a suitable exterior that will lead to a solution of the constraints.

5.1. Computing the parameter map. In this section we verify that the compactly supported deformation which puts the constraint data into the approximate kernel preserves the asymptotic symmetry conditions (AC) on the annulus. This will allow us to conclude that the projection of the data onto the cokernel is given to leading order by the change in parameters across the annulus from those of original data set to those of the model we glued on; i.e., the deformation does not induce any extra error terms. This simplifies the computations in [**C**], where less precise estimates were used along with explicit calculations in the conformally flat case; we note that a similar analysis could be carried out here.

5.1.1. Preservation of the AC Condition. Starting with g and π satisfying the asymptotic condition (AC) (in an AF chart, including several derivatives), we glue on a model solution on A_R, using a symmetric cut-off function which is zero near one of the boundary spheres, and one near the other; this produces an approximate solution $(\widetilde{g}, \widetilde{\pi})$. We note that the model solutions which we consider satisfy (AC), uniformly near a given one. Next, we pullback and scale to the unit annulus A_1: $(g_R)_{ij}(x) := \widetilde{g}_{ij}(Rx)$, and $(\pi_R)_{ij}(x) := R\widetilde{\pi}_{ij}(Rx)$. We see that $(g_R)_{ij}(x)-(g_R)_{ij}(-x) = O(R^{-2})$, $(\pi_R)_{ij}(x)+(\pi_R)_{ij}(-x) = O(R^{-2})$, and similarly the derivatives will satisfy the appropriate even/odd condition to order $O(R^{-2})$, and hence the Christoffel symbols and curvatures do as well. For a k-tensor T we let $\widehat{T} = (-1)^k\alpha^*T$, where α is the antipodal map; note that if T is a $(0,2)$-tensor, then $\widehat{T}_{ij}(x) = T_{ij}(-x)$ (Euclidean coordinates). We also recall that "O" will include several derivatives of the quantity in question (as required in the next section), and on the annulus A_1 the derivatives will decay at the same rate by the scaling.

Let $(h,\omega) = \rho D\Phi^*_{(g_R,\pi_R)}(u,Z)$ solve the nonlinear projected problem; in particular, (h,ω) is obtained by iteration, so that $(h,\omega) = \sum_{k=0}^{\infty}(h_k,\omega_k)$, where $(u,Z) = \sum_{k=0}^{\infty}(u_k,Z_k)$ and $(h_k,\omega_k) = \rho D\Phi^*_{(g_R,\pi_R)}(u_k,Z_k)$, and

$$\Phi(g_R+h,\pi_R+\omega) = \left(\sum_{i=0}^{3} c_i x^i\zeta, \sum_{j=1}^{3} a_j X_j\zeta + \sum_{k=1}^{3} b_k Y_k\zeta\right)$$

on A_1, with $x^0 := 1$. Since $\Phi(g_R,\pi_R) = O(R^{-1})$, we have by (26) that $(h,\omega) = O(R^{-1})$.

Lemma 5.1. *$(h-\widehat{h},\omega+\widehat{\omega}) = O(R^{-2})$.*

Proof. We find that the anti-symmetric part of $\Phi(g_R+h,\pi_R+\omega)$ is relatively small. Indeed, for $i=1,2,3$ we have $c_i = O(R^{-2})$, $b_i = O(R^{-2})$:

$$\begin{aligned}
&\int_{A_1}\mathcal{H}(g_R+h,\pi_R+\omega)x^i\,dx\\
&=\int_{A_1}\big[R(g_R)+L_{g_R}(h)+O(\|h\|^2_{2,\alpha})+O(R^{-2})\big]x^i\,dx\\
&=\int_{A_1\cap\{x^i>0\}}\big[R(g_R)-\widehat{R(g_R)}\big]x^i\,dx+\int_{A_1}L_{g_R}(h)x^i d\mu_{g_R}+O(R^{-2})
\end{aligned}$$

where we have used the fact that $|dx-d\mu_{g_R}| = O(R^{-1})$. The first term above is $O(R^{-2})$ by the symmetry of $R(g_R)$, as is the second integral, since $L^*_{g_R}(x^i) = O(R^{-1})$; indeed we note

$$\big(\mathrm{Hess}_{g_R}x^i\big)_{jk} = -\Gamma^m_{jk}(dx^i)_m = O(R^{-1}).$$

Similarly, integration by parts yields

$$\begin{aligned}\int_{A_1} \operatorname{div}_{g_R+h}(\pi_R+\omega)\cdot_\delta Y_i\,dx &= \int_{A_1} \operatorname{div}_\delta(\pi_R+\omega)\cdot_\delta Y_i\,dx + O(R^{-2})\\ &= O(R^{-2}).\end{aligned}$$

It directly follows that $D\Phi_{(g_R,\,\pi_R)}\,(h,\,\omega) - \big(\widehat{D\,\Phi_{(g_R,\,\pi_R)}}(h,\,\omega)\big) = \rho^{1/2}O(R^{-2})$. With a little computation, we also see that $D\Phi_{(g_R,\pi_R)}(h-\widehat{h},\omega+\widehat{\omega})$ and $D\Phi_{(g_R,\pi_R)}\rho D\Phi^*_{(g_R,\pi_R)}(u-\widehat{u},Z-\widehat{Z})$ are also $O(R^{-2})$. Here we use the fact that we have uniform bounds on $\rho\nabla^k u$ $(k=0,1,2)$ and $\rho\nabla^k Z$ $(k=0,1)$ and their derivatives from the elliptic estimates (and the decay of ρ).

Now we estimate $(h-\widehat{h},\omega+\widehat{\omega})=\rho\Big(D\Phi^*_{(g_R,\pi_R)}(u,Z)-D\Phi^*_{(g_R,\pi_R)}\widehat{(u,Z)}\Big)$ $+O(R^{-2})$. By the elliptic estimates of Section 4.3, using the interior estimate near the boundary to establish decay and global bounds, we see that it suffices to show that $(u,Z)-\widehat{(u,Z)}$ is $O(R^{-2})$ in $L^2_\rho(A_1)$. To show this, we note that the above symmetry discussion, coupled with linearization and the decay estimates obtained from the elliptic estimates in Section 4.3, shows that $D\Phi_{(g_R,\pi_R)}\rho D\Phi^*_{(g_R,\pi_R)}(u-\widehat{u},Z-\widehat{Z})\in L^2_{\rho^{-1}}(A_1)$ and that we have by Cauchy-Schwarz (and integration by parts, to handle the difference between the terms $D\Phi_{(g_R,\pi_R)}(h,\omega)-\big(\widehat{D\Phi_{(g_R,\pi_R)}}(h,\omega)\big)$ and $D\Phi_{(g_R,\pi_R)}\rho D\Phi^*_{(g_R,\pi_R)}(u-\widehat{u},Z-\widehat{Z})$)

$$\begin{aligned}&\int_{A_1}\big((u,Z)-\widehat{(u,Z)}\big)D\Phi_{(g_R,\pi_R)}\rho D\Phi^*_{(g_R,\pi_R)}(u-\widehat{u},Z-\widehat{Z})d\mu_{g_R}\\ &\le CR^{-2}\|(u,Z)-\widehat{(u,Z)}\|_{H^{2,1}_\rho(A_1)}.\end{aligned}$$

Furthermore, (u,Z) and hence $\widehat{(u,Z)}$ are transverse to ζK. This allows us to apply integration by parts (using Lemma 2.1) and the basic elliptic estimate transverse to the cokernel (Theorem 3) to get

$$\begin{aligned}&\int_{A_1}\big((u,Z)-\widehat{(u,Z)}\big)D\Phi_{(g_R,\pi_R)}\rho D\Phi^*_{(g_R,\pi_R)}(u-\widehat{u},Z-\widehat{Z})d\mu_{g_R}\\ &=\int_{A_1}\rho|D\Phi^*_{(g_R,\pi_R)}(u-\widehat{u},Z-\widehat{Z})|^2d\mu_{g_R}\\ &\ge C\|(u,Z)-\widehat{(u,Z)}\|^2_{H^{2,1}_\rho(A_1)}.\end{aligned}$$

q.e.d.

5.1.2. Controlling the resulting constraint data. The preservation of the (AC) condition implies that the projection of the constraint functions in the direction of the cokernel after solving the nonlinear projected problem is given, up to lower-order error terms, by boundary integrals; of course the boundary data *and its derivatives* are unchanged by the perturbation in the annulus. In particular, this means that the constraint functions can be controlled by varying the parameters of the solution we glue on, and so as long as the family we glue on has enough degrees of freedom, we can make the Hamiltonian and momentum constraints zero. Indeed, we let $(\overline{g}_R, \overline{\pi}_R) = (g_R + h, \pi_R + \omega)$, we let $(\overline{g}, \overline{\pi})$ be the re-scaled version on A_R, and we note for $k = 1, 2, 3$

$$\begin{aligned}
\int_{A_1} x^k \mathcal{H}(\overline{g}_R, \overline{\pi}_R) dx &= \int_{A_1} x^k \sum_{i,j} \big[(\overline{g}_R)_{ij,ij} - (\overline{g}_R)_{ii,jj}\big] dx + O(R^{-3}) \\
&= R^{-2} \int_{A_R} x^k \sum_{i,j} \big(\overline{g}_{ij,ij} - \overline{g}_{ii,jj}\big) dx + O(R^{-3}) \\
&= R^{-2} \oint_{\partial A_R} x^k \sum_{i,j} \big(\tilde{g}_{ij,i} - \tilde{g}_{ii,j}\big) \nu^j d\xi \\
&\quad - R^{-2} \oint_{\partial A_R} \sum_i \big(\tilde{g}_{ik} \nu^i - \tilde{g}_{ii} \nu^k\big) d\xi + O(R^{-3}).
\end{aligned}$$

Here we have used the fact that the difference between $R(\overline{g}_R)$ and $\sum_{i,j} \big[(\overline{g}_R)_{ij,ij} - (\overline{g}_R)_{ii,jj}\big]$ is $O(R^{-2})$ and is even to $O(R^{-3})$, and similarly for the quadratic terms in $\overline{\pi}_R$ in the Hamiltonian constraint. We now note the other projections, starting with the mass term, the projection onto the constant functions:

$$\begin{aligned}
\int_{A_1} \mathcal{H}(\overline{g}_R, \overline{\pi}_R) dx &= \int_{A_1} \sum_{i,j} \big[(\overline{g}_R)_{ij,ij} - (\overline{g}_R)_{ii,jj}\big] dx + O(R^{-2}) \\
&= R^{-1} \int_{A_R} \sum_{i,j} \big(\overline{g}_{ij,ij} - \overline{g}_{ii,jj}\big) dx + O(R^{-2}) \\
&= R^{-1} \oint_{\partial A_R} \sum_{i,j} \big(\tilde{g}_{ij,i} - \tilde{g}_{ii,j}\big) \nu^j d\xi + O(R^{-2}).
\end{aligned}$$

Integrating the momentum constraint against the translation fields X_k yields

$$\begin{aligned}
\int_{A_1} \operatorname{div}_{\overline{g}_R}(\overline{\pi}_R) \cdot X_k \, dx &= \int_{A_1} \operatorname{div}_{\delta}(\overline{\pi}_R) \cdot X_k \, dx + O(R^{-2}) \\
&= \oint_{\partial A_1} (\overline{\pi}_R)_{ij} \cdot X_k^i \nu^j \, d\xi + O(R^{-2}) \\
&= R^{-1} \oint_{\partial A_R} \overline{\pi}_{ij} X_k^i \nu^j \, d\xi + O(R^{-2}).
\end{aligned}$$

Finally, we project the momentum constraint onto the rotation fields Y_k:

$$\begin{aligned}\int_{A_1} \operatorname{div}_{\overline{g}_R}(\overline{\pi}_R)\cdot Y_k\,dx &= \int_{A_1}\operatorname{div}_{\delta}(\overline{\pi}_R)\cdot Y_k\,dx + O(R^{-3})\\ &= \oint_{\partial A_1}(\overline{\pi}_R)_{ij}\cdot Y_k^i\nu^j\,d\xi + O(R^{-3})\\ &= R^{-2}\oint_{\partial A_R}\overline{\pi}_{ij}Y_k^i\nu^j\,d\xi + O(R^{-3}).\end{aligned}$$

Note that we could also have used the measures from the metric $\overline{g}_R$ in these integrals (cf. Eqs. (1)–(4)). Also recall that $(\overline{g},\overline{\pi})$ agree with the given data (g,π) on the inner boundary, and agree with the data of the asymptotic model on the outer boundary of the annulus. To show the map $\mathcal{I}$ has a zero, we show the associated map $\mathcal{I}_R : \lambda \mapsto$

$$\Bigg(R\int_{A_1}\mathcal{H}(\overline{g}_R,\overline{\pi}_R)\,dx,\quad R\int_{A_1}\operatorname{div}_{\overline{g}_R}(\overline{\pi}_R)\cdot X_k\,dx,\\ R^2\int_{A_1}\operatorname{div}_{\overline{g}_R}(\overline{\pi}_R)\cdot Y_k\,dx,\quad R^2\int_{A_1}x^k\mathcal{H}(\overline{g}_R,\overline{\pi}_R)dx\Bigg)$$

has a zero.

5.2. Solving the constraints. We now finish the proof of Theorem 4. Thus far, we have not chosen which member of the model family (e.g., which slice in which Kerr) to use near infinity. We understand that the model will have ADM energy-momentum, and angular and linear momentum, near that of the original data. Suppose $\lambda_0 \in \mathcal{O}$ parametrizes a solution whose energy-momenta $\Theta(\lambda_0)$ agrees with that of (g,π). We then consider λ near λ_0.

The computation in the previous section shows then that the map $\mathcal{I}_R(\lambda) = \Theta(\lambda) - \Theta(\lambda_0) + o(1)$, where $o(1) \to 0$ as $R \to \infty$, uniformly for λ near λ_0. Fix a small ball B about λ_0. Let $\mathcal{I}_R(t,\lambda) = \Theta(\lambda) - \Theta(\lambda_0) + to(1)$ be a homotopy defined on $[0,1]\times\overline{B}$, between the homeomorphism $\Theta(\lambda) - \Theta(\lambda_0)$ and the map $\mathcal{I}_R$. For sufficiently large R, we have $0 \notin \mathcal{I}_R([0,1]\times\partial B)$. By degree considerations [**N**], for large R, $\mathcal{I}_R$ must hit zero for some $\lambda \in B$. For such a λ, the constraints are satisfied, with the model near infinity corresponding to λ for which $\mathcal{I}_R(\lambda) = 0$, and with $\Theta(\lambda)$ near $\Theta(\lambda_0)$. q.e.d.

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